On the equilibrium problem for masonry beams

Massimiliano Lucchesi[1], Miroslav Šilhavý[2], Nicola Zani[1]

[1] Dipartimento di Costruzioni, Università di Firenze, Italy
massimiliano.lucchesi@unifi.it nicola.zani@unifi.it

[2] Institute of Mathematics the AV CR, Prague, Czech Republic
silhavy@math.cas.cz

In this paper we consider beams made of a no-tension material with infinite compressive strength according to the constitutive equation described in [4] and [5]. This equation expresses the (generalized) stresses $\boldsymbol{t} = (N, M)$ as functions of the (generalized) strains $\boldsymbol{e} = (\epsilon, \kappa)$, with N the normal force, M the bending moment, ϵ the (linearized) strain and κ the change of curvature of the axis of the beam.

We suppose that the displacement fields $\boldsymbol{u} = (w, v)$ with w and v the axial and the transversal components, belong to the Sobolev space $W = W^{1,2} \times W^{2,2}$ and denote by V the (closed) subspace of W made of all the displacement fields that satisfy the boundary conditions. We put $Y = L^2(I, \mathbb{R}^2)$, $Y_0 = \{\widehat{\boldsymbol{e}}(\boldsymbol{u}) : \boldsymbol{u} \in V\}$ and note that $Y_0 \subset Y$. The loads applied to the beam are specified by a continuous linear functional $\boldsymbol{l} : V \to \mathbb{R}$ and then the value $< \boldsymbol{l}, \boldsymbol{u} >$ of $\boldsymbol{l}$ on $\boldsymbol{u}$ is the work of the load $\boldsymbol{l}$ corresponding to the admissible displacement $\boldsymbol{u} \in V$. We denote by $\widehat{\phi}$ the stored energy and prove that it is strictly convex in stresses and bounded by some quadratic function of $|\boldsymbol{e}|$. Moreover, we say that $\boldsymbol{u}_0 \in V$ is an equilibrium state if it minimizes the total energy E.

Firstly we prove that there exists a stressfield equilibrating the loads if and only if we have $< \boldsymbol{l}, \boldsymbol{u} > \ \le c\, \|\widehat{\boldsymbol{e}}(\boldsymbol{u})\|_Y$ for each $\boldsymbol{u} \in V$ and some $c \in \mathbb{R}$.

Then we prove that if there exists a strain $\boldsymbol{e} \in Y_0$ and a stressfield $\boldsymbol{t} \in Y$ equilibrating the loads $\boldsymbol{l}$ such that $\mathrm{D}\widehat{\phi} \circ \boldsymbol{e} = \boldsymbol{t}$, then the external conditions have an equilibrium state and $\boldsymbol{t}$ is admissible (i.e. $2|M/N|$ is less than the height of the section). Conversely, if $\boldsymbol{u}_0$ is an equilibrium state, then $\boldsymbol{t} = \mathrm{D}\widehat{\phi} \circ \widehat{\boldsymbol{e}}(\boldsymbol{u}_0) \in Y$ is an admissible stressfield that equilibrates the loads. Moreover, if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two equilibrium states for the loads $\boldsymbol{l}$ and $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ the stressfield corresponding to them, then $\boldsymbol{t}_1 = \boldsymbol{t}_2$.

Finally, we prove a proposition which states that the minimum of the total energy $E_0 : = \inf\{E(\boldsymbol{u}) : \boldsymbol{u} \in V\}$ is bounded from below if and only if there exists an admissible stressfield $\boldsymbol{t} \in Y$ which equilibrates the loads. This last result is important in the study of limit analysis of masonry structures where the loads are the sum of a permanent and variable part, and depend linearly on a parameter λ, $\boldsymbol{l}(\lambda) = \boldsymbol{l}_0 + \lambda \boldsymbol{l}_1$. Indeed the supremum of the interval where $E_0(\lambda)$ is bounded from below can be interpreted as the collapse multiplier and then from this the static theorem of limit analysis follows easily [1, 2, 3].

Acknowledgment The research of M. Šilhavý was supported by Dipartimento di Costruzioni, Università di Firenze. The support is gratefully acknowledged.

References

1 Del Piero, G. (1998) Limit analysis and no-tension materials, *Int. J. Plasticity* **14** 259-271

2 Lucchesi, M.; Padovani, C.; Šilhavý, M. (2009) An energetic view of the limit analysis for normal bodies. To appear in *Quarterly of Applied Mathematics*

5 Lucchesi, M.; Šilhavý, M., Zani N. (2010) Equilibrium problems and limit analysis of

masonry beams. Submitted.

Orlandi, D. (1999) Analisi non lineare di strutture ad arco in muratura, Ph.D. thesis, Università degli studi di Firenze

Zani, N. (2004) A constitutive equation and a closed-form solution for no-tension beams with limited compressive strength. *European J. Mechanics A/Solids* **23** 467-484