On uniqueness of solutions of frictional contact problems in linear elasticity.

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The problem of the equilibrium of a linearly elastic body above a rigid obstacle in the frictionless situation (the so-called Signorini) admits a unique solution, as proved in the sixties by G. Fichera, G. Stampacchia and J.L. Lions. In the early beginning of the seventies, G. Duvaut and J.L. Lions considered the more general situation where the contact condition is complemented with Coulomb friction law:

$$\forall\, \mathbf{v}, \qquad \mathbf{t}_t \cdot \big(\mathbf{v} - \dot{\mathbf{u}}_t\big) - \mathcal{F} t_n \big(|\mathbf{v}| - |\dot{\mathbf{u}}_t|\big) \geq 0, \tag{1}$$

where $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ is the surface traction, $\mathbf{t} = \mathbf{t}_t + t_n \mathbf{n}$ its splitting into tangential and normal parts, $\dot{\mathbf{u}}$ the velocity of material points at the boundary and $\mathcal{F}$ a given friction coefficient. The Coulomb law (1) is the simplest phenomenological law that describes the occurrence of dry friction. Because this law is expressed in terms of the velocity, the corresponding contact problem becomes an *evolution* problem, which is sometimes called the Signorini problem with quasi-static Coulomb friction. The analysis of that problem turned out to be very challenging and the coupling of linear elasticity and dry friction is still not understood nowadays. Many efforts have been devoted to the situation where the tangential velocity in the friction law (1) is replaced by the tangential displacement. The corresponding law is usually called the "static Coulomb friction law", although it may not be a particularly appropriate term. The reason for studying the contact problem with static Coulomb friction law is that this is formally the problem that arises at each time step when a time-discretization procedure is introduced into the analysis of the contact problem with quasi-static Coulomb friction.

It is known [Hild, 2004] that, for large friction coefficient $\mathcal{F}$, the contact problem with static Coulomb friction may have multiple solutions. One crucial question which is still open today is to know whether uniqueness of solutions holds true for sufficiently small friction coefficient $\mathcal{F} < \mathcal{F}_{\mathrm{c}}$, or not.

In this talk, this question of uniqueness will be adressed in two specific cases.

1. The indentation of a bidimensional isotropic elastic half-space by a rigid flat punch. It is proved that the border between sticking and slipping zones on the boundary is "frank". Asymptotic estimates for the surface tractions and displacement in the vicinity of such a border are exhibited. Partial uniqueness results are given.

2. The case of a bounded body of arbitrary geometry in (bidimensional) incompressible isotropic elasticity. A complete analysis of existence and uniqueness of solution is provided in this favourable background.

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