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A survey of variational and numerical formulations using bipotentials in mechanics of structures

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Fundamentals

Modelisation of constitutive laws
by a numerical function

Graph $M \subset X \times Y$

Potential $y = D\varphi(x)$ $\varphi(x)$ smooth and convex

Cyclically monotone maximal graph

Superpotential $y \in \partial\varphi(x)$ $\varphi(x)$ convex **J.J. Moreau(1963)**

Monotone maximal graph

Fitzpatrick's function $F(x, y)$ globally convex **S. Fitzpatrick (1988)**

Selfdual lagrangian $y \in \partial L(., y)(x)$ $L(x, y)$ globally convex **N. Ghoussoub(2007)**

"BB-graph"

Bipotential $y \in \partial b(., y)(x)$ $b(x, y)$ biconvex

G. de Saxcé, Z.-Q. Feng(1991)

The bipotential in short

de Saxcé, Feng(1991) **Buliga, de Saxcé et al. (08)**

bipotential $(x,y) \mapsto b(x,y) \in IR \cup \{-\infty, +\infty\}$ biconvex, bi-l.s.c., s.t.

cornerstone inequality $\forall (x', y') \quad b(x', y') \geq \langle x', y' \rangle$

generalizes Fenchel's inequality

$\varphi(x') + \varphi^{*}(y') \geq \langle x', y' \rangle$

for (x, y) there is equivalence $b(x,y) = \langle x, y \rangle$

$\Updownarrow$

Implicit sub-normality $y \in \partial b(., y)(x)$ constitutive law

$\Updownarrow$

$x \in \partial b(x, .)(y)$ inverse law

Typology of bipotentials

Bipotential representing a non associative constitutive law

friction contact

- ✓ Coulomb's law — **de Saxcé, Feng (1991)**
- ✓ Mróz-Michałowski orthotropic friction — **de Saxcé, Hjiaj et al.(2004)**

soils mechanics models

- ✓ non associated Drucker-Prager — **Berga, Bousshine, de Saxcé(1992)**
- ✓ modified cam-clay — **N. Zouian et al.(IJSS, 2007)**

non linear hardening rule in cyclic plasticity — **de Saxcé(1992)**

Lemaitre's plastic ductile damage — **G. Bodovillé (CRAS,1999)**

endochronic theory — **S. Erlicher, N. Point(2006)**

Bipotentials representing a family of constitutive laws

coaxial laws:

- ✓ Cauchy's bipotential $b(x,y) = \|x\| \, \|y\|$ for the law $y = \lambda\, x, \quad \lambda \geq 0$
- ✓ Hill bipotential $b(x,y) = \sum_{i=1}^{n} \lambda_i(x) \lambda_i(y)$ — **Vallée et al. (1997)**

Fundamentals

Existence and (non) unicity of a bipotential

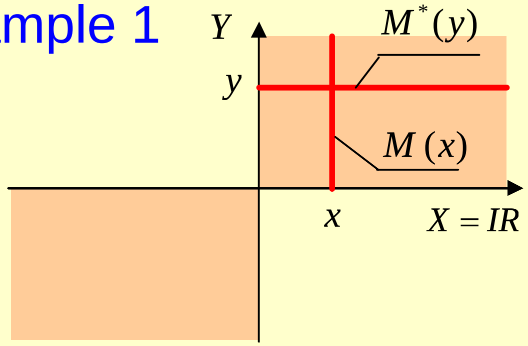
Theorem *I* "BB-graph"

a NSC for a graph $M \subset X \times Y$ admitting a bipotential b: $M =: \{(x,y) \quad \text{s.t.} \quad b(x,y) = \langle x,y \rangle \}$
is that all sections $x = C^{te}$ and $y = C^{te}$ are convex and closed

Proof: $b_\infty(x,y) = \langle x,y \rangle \quad \text{if} \quad (x,y) \in M$

$= +\infty \qquad \text{otherwise}$

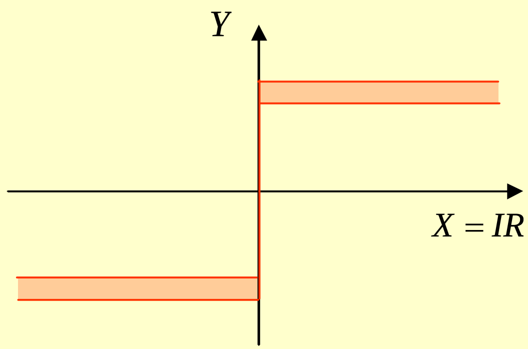
Example 1



Example 2

"blurred law"

Buliga, de Saxcé, Vallée (MMS,2009)



Remark: generally speaking, the bipotential is not unique

Example : the graph of the coaxial laws admits $\|x\| \, \|y\|$ and $b_\infty(x,y)$

New question: for M being given, how to construct a bipotential which is not +8 outside M?

Interest of a systematic construction

It is not necessary to be extremely skilful in handling inequalities to find a bipotential

An algorithm of construction of b

"bipotential convex cover" M. Buliga, G. de Saxcé, C. Vallée (JCA,08-09)

1) covering M by c.m. sub-graphs M_λ

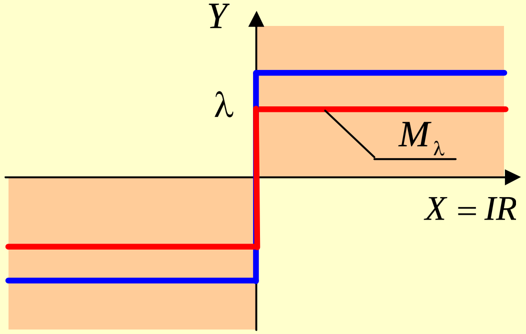
2) calculating $x \rightarrow \varphi_\lambda(x)$ and $y \rightarrow \chi_\lambda(y)$ by Rockafellar's theorem

3) constructing the bipotentials

$$b_\lambda(x,y) = \sup\left(\varphi_\lambda(x) + \varphi_\lambda^*(y)\, ,\, \chi_\lambda^*(x) + \chi_\lambda(y)\right)$$

4) constructing the bipotential

$$b(x,y) = \inf_{\lambda \in \Lambda} b_\lambda(x,y)$$

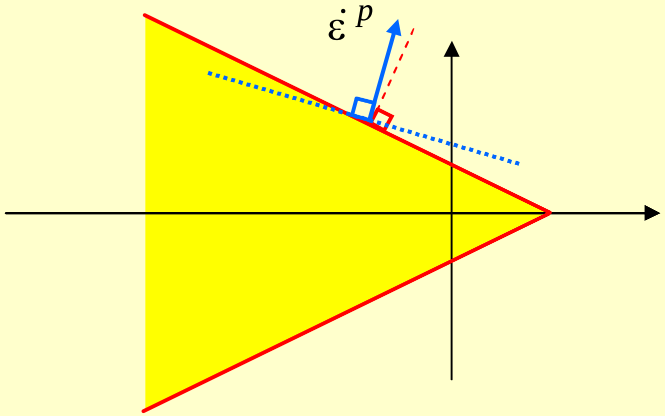


Example > For Coulomb's dry friction, with this algorithm

$$b(x,y) = \mu\, y_n \|x_t\| \quad \text{if} \quad x_n \le 0 \ \text{and} \ \|y_t\| \le \mu\, y_n$$
$$= +\infty \qquad \text{otherwise}$$

Application to Soil Plasticity

soil mechanics



Incremental bipotential $\Delta b(\Delta\varepsilon, \Delta\sigma) = \inf_{\Delta\varepsilon_p} \left(\Delta b_e(\Delta\varepsilon - \Delta\varepsilon_p, \Delta\sigma) + \Delta b_p(\Delta\varepsilon_p, \Delta\sigma)\right)$

Elastoplastic law $\Delta\sigma_r = \dfrac{\partial \Delta b}{\partial \Delta\varepsilon}(\Delta\varepsilon, \Delta\sigma) - \Delta\sigma = 0$

local tangent matrix $K_t = \left(\dfrac{\partial \Delta b}{\partial \Delta\sigma\, \partial \Delta\varepsilon} - I\right)^{-1} \boxed{\dfrac{\partial^2 \Delta b}{\partial \Delta\varepsilon^2}}$ hessian matrix

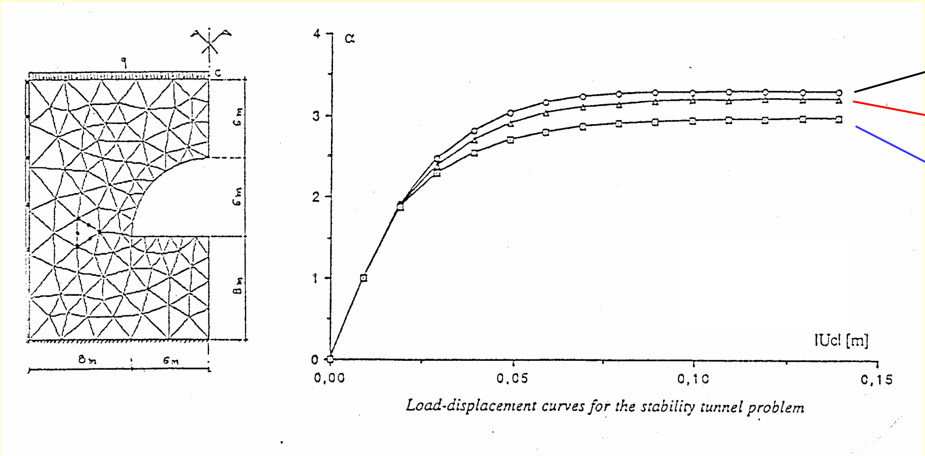
symmetric

positive

modified Newton method $K_b = \dfrac{\partial^2 \Delta b}{\partial \Delta\varepsilon^2}$

A. Berga, G. de Saxcé(1994)

Application to Soil Plasticity



$\varphi = \theta = 20°$

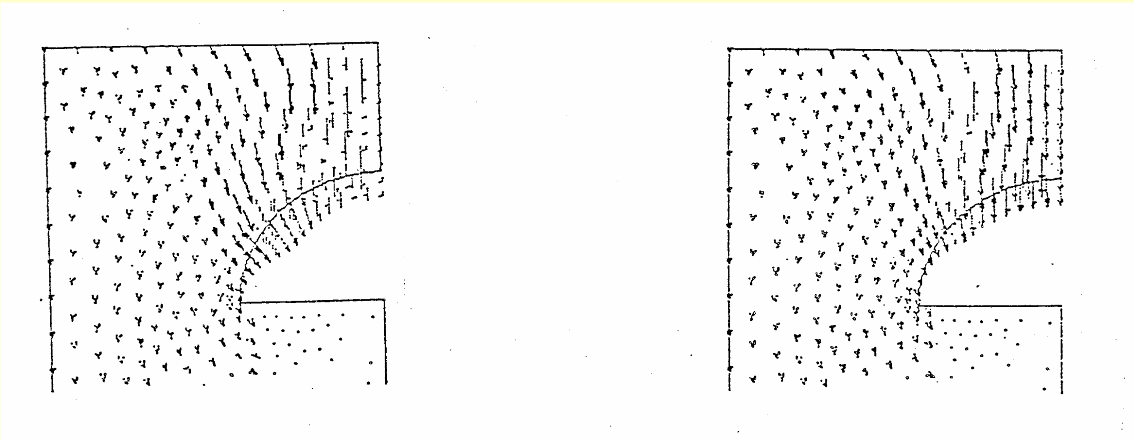
$\theta = 10°$

$\theta = 0°$

Berga, de Saxcé(1994)

gain in computation time > 30 %

Collapse mechanism $\varphi = 20°$



associated law ($\theta = 20°$)

non associated law ($\theta = 10°$)

Applications

Variational approaches

bifunctional $$B(\dot{u},\sigma) = \int_\Omega b(\varepsilon(\dot{u}),\sigma)\, d\Omega - \int_{\Gamma_0} p(\sigma)\, \dot{u}_d\, d\Gamma - \int_{\Gamma_1} p_d\, \dot{u}\, d\Gamma - \int_\Omega f_d\, \dot{u}\, d\Omega$$

cornerstone inequality of the bipotential

Green's formula

$\Rightarrow$ $\forall \dot{u}'\ K.A.$, $\forall \sigma'\ S.A.$ $\quad B(\dot{u}',\sigma') \geq 0$

for the solutions of the boundary value problem $\quad B(\dot{u},\sigma) = 0$

iterative algorithm

Bousshine, de Saxcé(1992)

do

update the velocity field $\dot{u}_{(i+1)} = \arg\inf \{ B(\dot{u}',\sigma_{(i)}) : \dot{u}'\ K.A. \}$ global phase

update the stress field $\sigma_{(i+1)} \in \partial b(\dot{u}_{(i+1)},\sigma_{(i)})$ local phase

until convergence

Applications

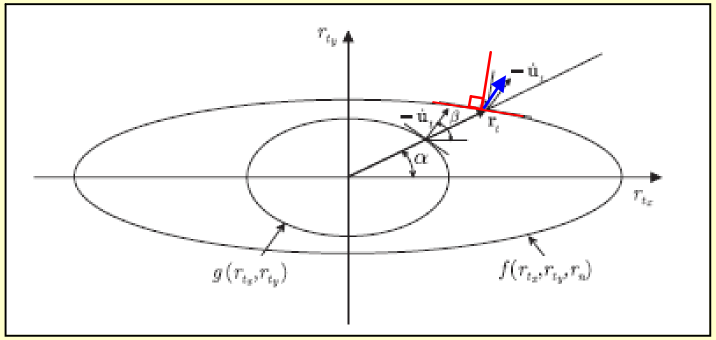
Modelling the orthotropic friction

Z. Mróz, R. Michałowski (1978)

friction cone K_μ $$f(r) = \sqrt{(r_{tx}/\mu_x)^2 + (r_{ty}/\mu_y)^2} - r_n \le 0$$

sliding potential $$g(r) = \sqrt{(r_{tx}/p_x)^2 + (r_{ty}/p_y)^2}$$

$$-\dot{u}_t = \dot{\lambda}\frac{\partial g}{\partial r_t}$$



M. Hjiaj, Z.Q. Feng, G. de Saxcé, Z. Mróz (IJNME, 2004)

non-associativity matrix $$Q = diag\,(p_x/\mu_x, p_y/\mu_y)$$

reformulating the law $$T(\dot{u}) = Q^2\dot{u}_t - (\dot{u}_n + \left\|Q^2\dot{u}_t\right\|_\mu^*)\,n$$

$$T(-\dot{u}) \in \partial\chi_{K_\mu}(r)$$

contact bipotential $$b_c(-\dot{u}, r) = \begin{cases} \langle(-\dot{u}) - T(-\dot{u}), r\rangle & if \quad r \in K_\mu \quad and \quad -\dot{u}_n \le 0 \\ +\infty & otherwise \end{cases}$$

Building a numerical scheme

incremental problem

Find Δr s.t.

$$-\Delta u \in \partial \Delta b_c(-\Delta u,.)(\Delta r)$$

numerical parameter $\rho > 0$

Find r_1 s.t.

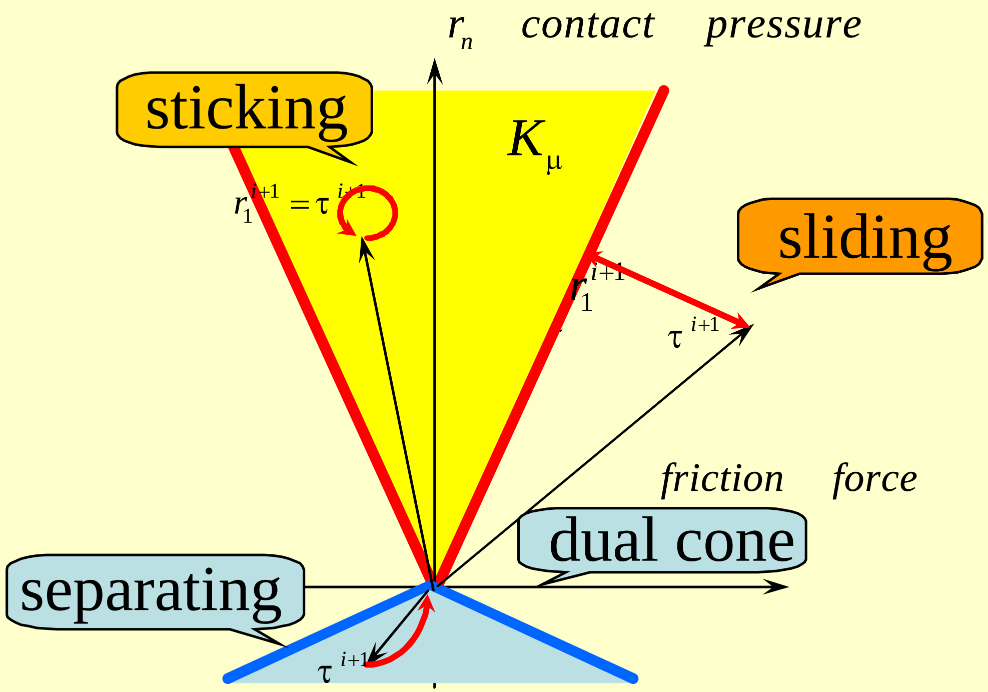
$$r_1 = proj\,(r_1 + \rho\, T\,(-\Delta u), K_\mu)$$

a predictor-corrector scheme
with projection

- ✓ predictor
- ✓ corrector

$$\tau^{\,i+1} = r_1^i + \rho\, T\,(-\Delta u^i)$$

$$r_1^{i+1} = proj\,(\tau^{\,i+1}, K_\mu)$$



3 events:
separating, sliding, sticking

Applications

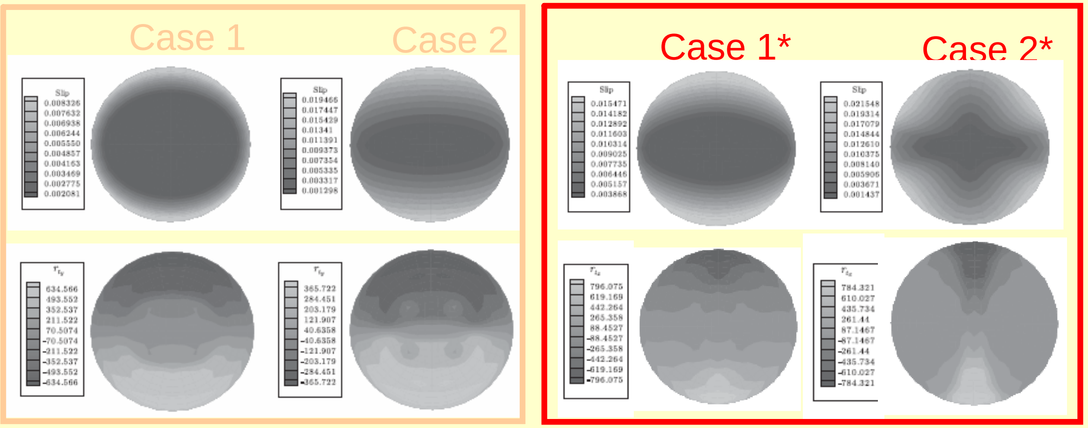
Numerical applications

Example 1 > compression load

associative non associative

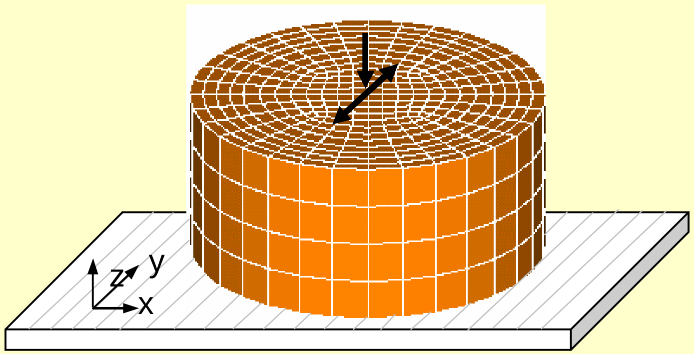
slip

friction stress

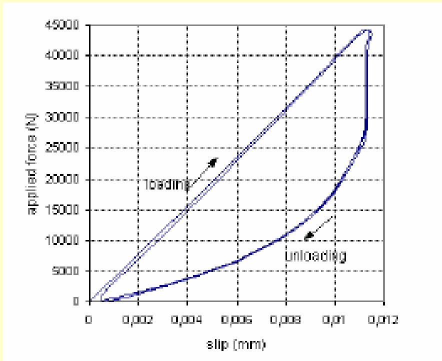


Example 2 > cyclic friction load

Z.Q. Feng et al.(IJNLM, 2006)



applied load



sliding

unlike the isotropic case, We observe a strong hysteretic behaviour

CONCLUSIONS

We established a existence criterion and
an algorithm of construction of a bipotential

We generalized the variational calculus with the bifunctional

We proposed a predictor/corrector scheme for frictional contact