# INTEGRATION OF MEASURES AND ADMISSIBLE STRESS FIELDS FOR MASONRY BODIES

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# Constitutive equations of no–tension materials

no–tension materials = masonry materials

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})$$

If $\boldsymbol{E} \in \mathrm{Sym}$, exists a unique triplet $(\boldsymbol{T}, \boldsymbol{E}^{e}, \boldsymbol{E}^{f})$ of elements of $\mathrm{Sym}$ such that

$$\left.\begin{array}{c} \boldsymbol{E} = \boldsymbol{E}^{e} + \boldsymbol{E}^{f}, \\ \boldsymbol{T} = \mathbb{C}\boldsymbol{E}^{e}, \\ \boldsymbol{T} \in \mathrm{Sym}^{-}, \quad \boldsymbol{E}^{f} \in \mathrm{Sym}^{+}, \\ \boldsymbol{T} \cdot \boldsymbol{E}^{f} = 0. \end{array}\right\}$$

***elastic stress*** $\hat{\boldsymbol{T}} : \mathrm{Sym} \to \mathrm{Sym}$ and ***stored energy*** $\hat{w} : \mathrm{Sym} \to \mathbf{R}$

$$\hat{\boldsymbol{T}}(\boldsymbol{E}) = \boldsymbol{T}, \quad \hat{w}(\boldsymbol{E}) = \frac{1}{2}\hat{\boldsymbol{T}}(\boldsymbol{E}) \cdot \boldsymbol{E}, \quad \hat{\boldsymbol{T}} = \mathrm{D}\,\hat{w}$$

[Del Piero, 1989], [Anzellotti, 1985], [Giaquinta and Giusti, 1985], [Di Pasquale, 1984], [Lucchesi et al., 1994]

# Loads and potential energy

the body

$$\Omega \subset \mathbf{R}^n$$

Lipschitz boundary with normal $\boldsymbol{n}$
loading

$$\partial\Omega = \mathcal{D} \cup \mathcal{S}$$

- on $\mathcal{D}$ prescribed the displacement $\boldsymbol{u} = \mathbf{0}$
- on $\mathcal{S}$ prescribed the surface traction $\boldsymbol{T}\boldsymbol{n} = \boldsymbol{s}$
- body force $\operatorname{div} \boldsymbol{T} + \boldsymbol{b} = \mathbf{0}$
- here $\boldsymbol{s}$ and $\boldsymbol{b}$ are given functions

## *total energy*

$$I(\boldsymbol{u}) = \int_{\Omega} \hat{w}(\hat{\boldsymbol{E}}(\boldsymbol{u}))\, d\mathscr{L}^n - \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{b}\, d\mathscr{L}^n - \int_{\mathscr{S}} \boldsymbol{u} \cdot \boldsymbol{s}\, d\mathscr{H}^{n-1},$$

## Equilibrium states

| |
|---|
| equilibrium states under given loads := absolute minimizers of total energy |

[Anzellotti, 1985], [Giaquinta and Giusti, 1985]:

- necessary to work in the space $BD(\Omega)$ of displacements of bounded deformation
- Neumann condition $\mathcal{D} = \emptyset$
- safe load condition: – there exists a stress field $T$ equilibrating the loads in the sense

$$\int_{\Omega} \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^n = \int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{v}\, d\mathcal{H}^{n-1} + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v}\, d\mathcal{L}^n$$

for all $\boldsymbol{v} \in W^{1,2}(\Omega, \mathbf{R}^n)$ such that

$$\boldsymbol{T}(\boldsymbol{x})\boldsymbol{a} \cdot \boldsymbol{a} \leq -\alpha|\boldsymbol{a}|^2$$

for some $\alpha > 0$, all $\boldsymbol{a} \in \mathbf{R}^n$ and almost all $\boldsymbol{x} \in \Omega$

Under these conditions,
there exists an equilibrium state corresponding to the given loads
stressfield is unique
dispalcement generally nonunique

- The applied loads are not arbitrary

# Lower bound for energy and strongly compatible loads

Assume

- $\boldsymbol{s} \in L^2(\mathscr{S})$, $\boldsymbol{b} \in L^2(\Omega)$
- $\boldsymbol{u} \in V := \{\boldsymbol{u} \in W^{1,2}(\Omega, \mathbf{R}^n) : \boldsymbol{u} = \mathbf{0} \text{ on } \mathscr{D}\}$

$$I(\boldsymbol{u}) = \int_\Omega \hat{w}(\hat{\boldsymbol{E}}(\boldsymbol{u}))\, d\mathscr{L}^n - \int_\Omega \boldsymbol{u} \cdot \boldsymbol{b}\, d\mathscr{L}^n - \int_{\mathscr{S}} \boldsymbol{u} \cdot \boldsymbol{s}\, d\mathscr{H}^{n-1},$$

$$I : V \to \mathbf{R}$$

$I$ is bounded from below on $V$ $\Leftrightarrow$ the loads are ***strongly compatible,*** i.e., there exists a $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym}^-)$ such that

$$\int_{\Omega} \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})\, d\mathcal{L}^n = \int_{\mathcal{S}} \boldsymbol{s} \cdot \boldsymbol{v}\, d\mathcal{H}^{n-1} + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v}\, d\mathcal{L}^n$$

for every $\boldsymbol{v} \in V$.

$$\inf \{I(\boldsymbol{u}) : \boldsymbol{u} \in V\} > -\infty$$ … the loads are safe

$$\inf \{I(\boldsymbol{u}) : \boldsymbol{u} \in V\} = -\infty$$ … collapse

*admissible equilibrating stress field* $\boldsymbol{T}$ $\Rightarrow$ a central problem of limit analysis [Del Piero, 1998], [Lucchesi et al., 2009], [Témam, 1983, Chapter 1, Section 5]

# Stresses represented by measures and weakly compatible loads

stresses represented by measures

$$\mathsf{T} \in \mathcal{M}(\Omega, \mathrm{Sym})$$

[tensor valued measure: countably additive set function on Borel sets of $\Omega$.]
If $\boldsymbol{T} : \Omega \to \mathrm{Sym}$ is an integrable function then

$$\mathsf{T}(A) = \int_{A} \boldsymbol{T}(\boldsymbol{x})\, d\mathcal{L}^{n}(\boldsymbol{x})$$

is a measure.

- stresses represented by a measure are generalizations of the stresses represented by functions
- $\Rightarrow$ allow for concentrations on surfaces and lines (and other lower dimensional objects)

The loads $(\boldsymbol{s}, \boldsymbol{b})$ are said to be ***weakly compatible*** if there exists a measure $\mathsf{T} \in \mathcal{M}(\Omega, \mathrm{Sym}^{-})$ such that

$$\int_{\Omega} \hat{\boldsymbol{E}}(\boldsymbol{u}) \cdot d\mathsf{T} = \int_{\mathcal{S}} \boldsymbol{s} \cdot \boldsymbol{u}\, d\mathcal{H}^{n-1} + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{u}\, d\mathcal{L}^{n}$$

for every

$$\boldsymbol{u} \in \left\{ \boldsymbol{u} \in C^{1}(\mathrm{cl}\,\Omega, \mathbf{R}^{n}), \boldsymbol{u} = \mathbf{0} \text{ on } \mathcal{D} \right\}.$$

- strongly compatible $\Rightarrow$ weakly compatible ***but not conversely***
- there are loads that are weakly compatible but not strongly compatible
- the loads can be weakly compatible and yet the total energy is not bounded from below

# Goal

describe a procedure which in certain cases allows us to use the information that the loads are weakly compatible to show that they are actually strongly compatible

# Affine loads and limit analysis

the functions $\boldsymbol{s}, \boldsymbol{b}$ depend on $\lambda$ linearly [Del Piero, 1998]. Thus $\mathfrak{L}^{\lambda} := (\boldsymbol{s}^{\lambda}, \boldsymbol{b}^{\lambda})$ where

$$\boldsymbol{s}^{\lambda} = \boldsymbol{s}_{\mathrm{O}} + \lambda \boldsymbol{s}_{1}, \quad \boldsymbol{b}^{\lambda} = \boldsymbol{b}_{\mathrm{O}} + \lambda \boldsymbol{b}_{1}, \quad \lambda \in \mathbf{R},$$

where

$$\boldsymbol{s}_{\mathrm{O}},\ \boldsymbol{s}_{1} \in L^{2}(\mathcal{S}, \mathbf{R}^{n}), \quad \boldsymbol{b}_{\mathrm{O}},\ \boldsymbol{b}_{1} \in L^{2}(\Omega, \mathbf{R}^{n}).$$

- $\boldsymbol{s}_{\mathrm{O}}, \boldsymbol{b}_{\mathrm{O}}$ the ***permanent part*** of the loads
- $\boldsymbol{s}_{1}, \boldsymbol{b}_{1}$ the ***variable part*** of the loads
- $\lambda$ the ***loading multiplier.***

to prove that the loads are strongly compatible if we know that they are weakly compatible

take the average of the stress measure over $(\mu - \varepsilon, \mu + \varepsilon)$ where $\varepsilon > 0$

$$\mathsf{T} = \frac{1}{2\varepsilon} \int_{\mu-\varepsilon,}^{\mu+\varepsilon} \mathsf{T}^{\lambda} d\lambda$$

# Families of measures

***integrable parametric measure***: a family $\{m^\lambda : \lambda \in \Lambda\}$ of measures on $\mathbf{R}^n$ where $\Lambda \subset \mathbf{R}$ is a $\mathscr{L}^1$ measurable set of parameters,

- measurable in $\lambda$
- finite total variation

parametric measures occur in the context of disintegration (slicing) of measures and in the context of Young's measures.

**Proposition.** *If* $\{m^\lambda : \lambda \in \Lambda\}$ *is an integrable parametric measure then there exists a unique measure* $m$ *on* $\mathbf{R}^n$ *such that*

$$\int_{\mathbf{R}^n} f \cdot dm = \int_{\Lambda} \int_{\mathbf{R}^n} f \cdot dm^\lambda \, d\lambda$$

*for each* $f \in C_0(\mathbf{R}^n, V)$.

We write

$$m = \int_{\Lambda} m^\lambda \, d\lambda$$

and call $m$ the ***integral of the family*** $\{m^\lambda : \lambda \in \Lambda\}$ with respect to $\lambda$.

**Proposition.** *Let* $\{h^\lambda : \lambda \in \Lambda\}$ *be a family of functions on* $\Omega \subset \mathbf{R}^n$

$$\int_{\Lambda}\int_{\Omega} |h^\lambda(\boldsymbol{x})| \, d\boldsymbol{x} d\lambda < \infty.$$

*If we define a measure* $m^\lambda$ *by*

$$m^\lambda = h^\lambda \mathscr{L}^n \llcorner \Omega$$

*then* $\{m^\lambda : \lambda \in \Lambda\}$ *is an integrable parametric measure and we have*

$$\int_{\Lambda} m^\lambda d\lambda = k\mathscr{L}^n \llcorner \Omega$$

*where*

$$k(\boldsymbol{x}) = \int_{\Lambda} h^\lambda(\boldsymbol{x}) \, d\lambda$$

*for* $\mathscr{L}^n$ *a.e.* $\boldsymbol{x} \in \Omega$.

**Proposition.** *Let* $\Omega_0 \subset \mathbf{R}^n$ *be open, let* $\varphi : \Omega_0 \to \mathbf{R}$ *be locally Lipschitz continuous and let* $g : \Omega_0 \to V$ *be* $\mathcal{L}^n$ *measurable on* $\Omega_0$*, with*

$$\int_{\Omega_0} |g||\nabla\varphi| \, d\mathcal{L}^n < \infty.$$

*Then for* $\mathcal{L}^1$ *a.e.* $\lambda \in \mathbf{R}$ *the function* $g$ *is* $\mathcal{H}^{n-1} \llcorner \varphi^{-1}(\lambda)$ *integrable; denoting by* $\Lambda$ *the set of all such* $\lambda$ *we define the measure* $m^\lambda$ *by*

$$m^\lambda := g\mathcal{H}^{n-1} \llcorner \varphi^{-1}(\lambda)$$

*for each* $\lambda \in \Lambda$. *Then* $\{m^\lambda : \lambda \in \Lambda\}$ *is an integrable parametric measure and we have*

$$\int_\Lambda m^\lambda \, d\lambda = g|\nabla\varphi|\mathcal{L}^n \llcorner \Omega_0.$$

# A panel under vertical top loads and horizontal side loads

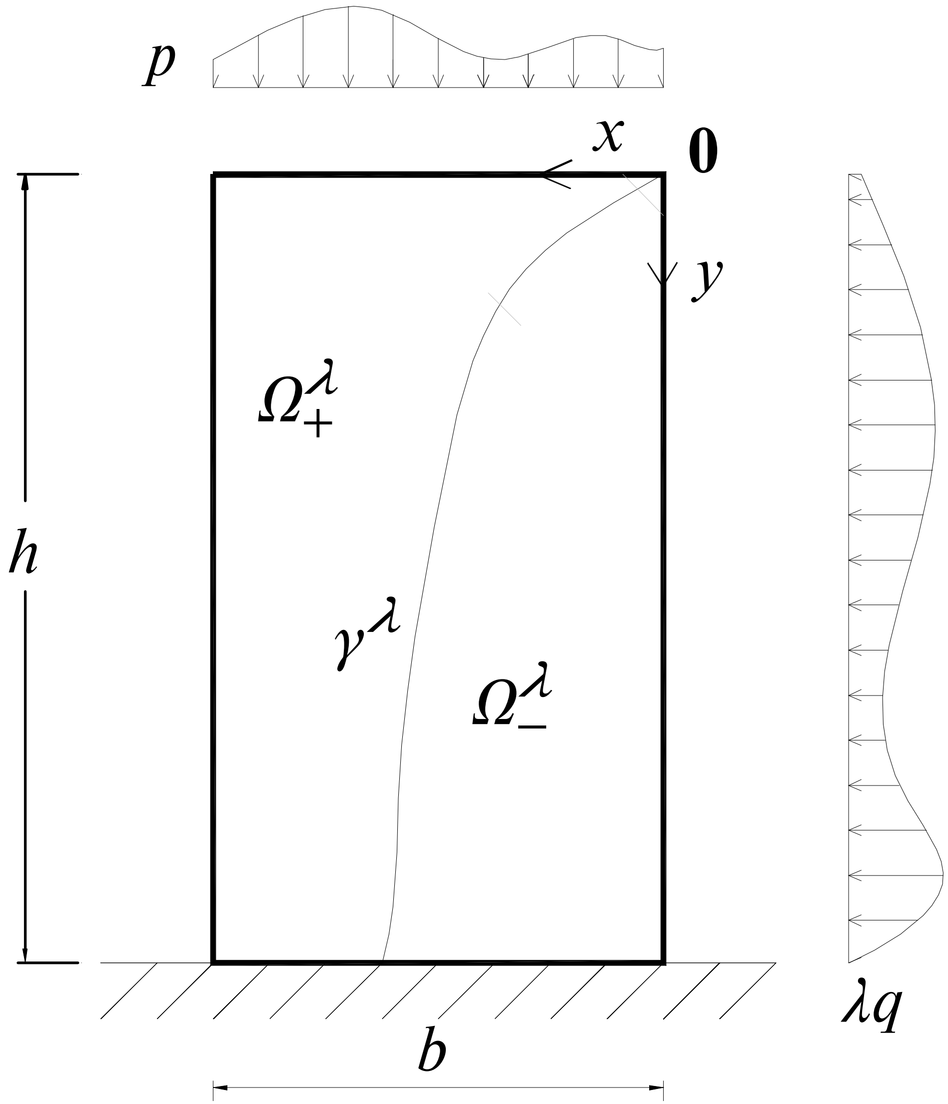
$$\Omega = (0,b) \times (0,h) \subset \mathbf{R}^2;$$

$$\mathcal{D} = (0,b) \times \{h\}, \quad \mathcal{S} = \partial\Omega \sim \mathcal{D}$$

and consider the loads $\mathfrak{L}^\lambda = (\boldsymbol{s}^\lambda, \boldsymbol{b}^\lambda)$ where $\boldsymbol{b}^\lambda = \mathbf{0}$ in $\Omega$, and, for $\boldsymbol{r} = (x,y) \in \mathcal{S}$,

$$\boldsymbol{s}^\lambda(\boldsymbol{r}) = \begin{cases} p(x)\boldsymbol{j} & \text{on} \quad (0,b) \times \{0\}, \\ \lambda q(y)\boldsymbol{i} & \text{on} \quad \{0\} \times (0,h), \\ \mathbf{0} & \text{elsewhere,} \end{cases}$$

where $p \geq 0, q \geq 0$




The panel under vertical top loads and horizontal side loads.

We assume that

$$p_0 := p(0) > 0, \quad q_0 := q(0) > 0.$$

There exists $\lambda_c > 0$ such that the loads are weakly compatible for $0 \leq \lambda \leq \lambda_c$. The corresponding stress is absolutely continuous in $\Omega$ except on a curve $\gamma^\lambda$ where it is absolutely continuous with respect to the length measure. The curve starts at the upper right corner of the panel and ends at the base of the panel.

**Proposition.** *If* $0 < \mu < \lambda_c$ *then the loads* $\mathfrak{L}^\mu$ *are strongly compatible.*

# Example: Explicit determination of the averaged stress field

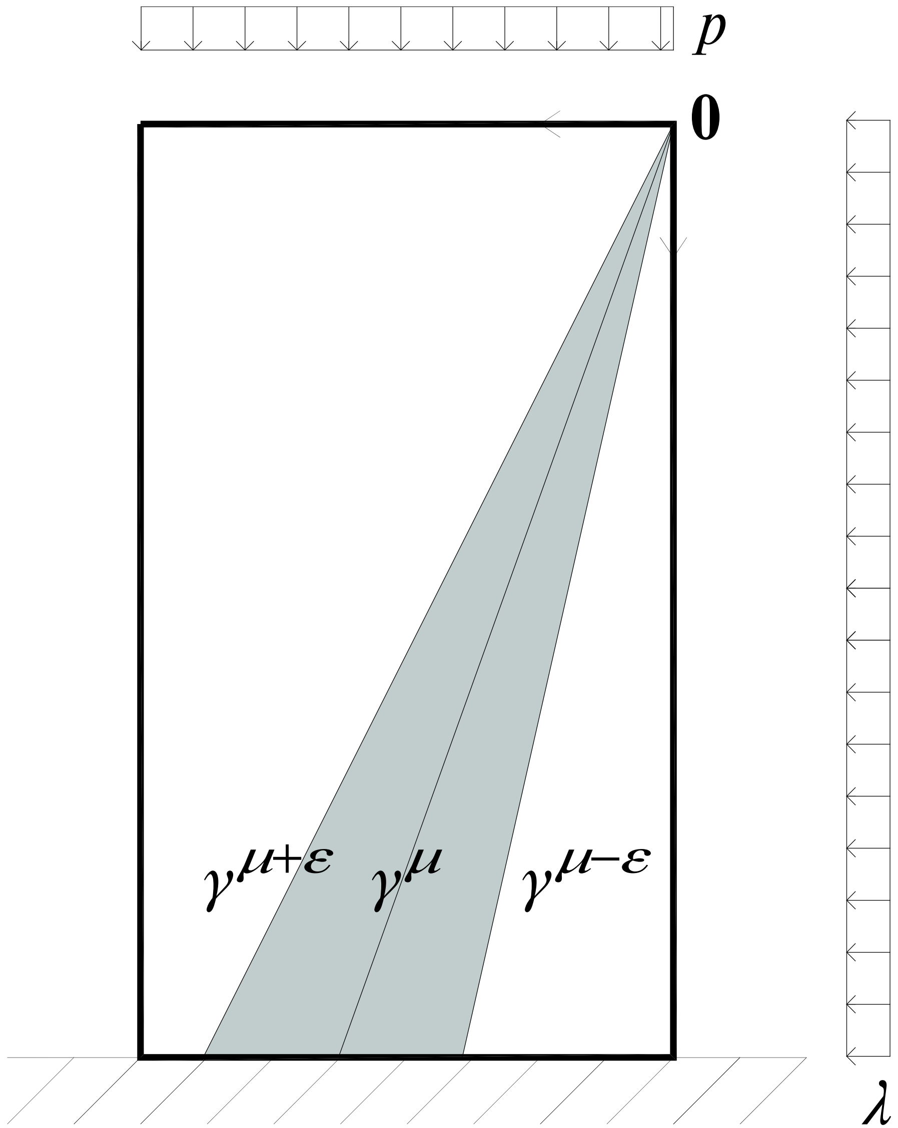
in particular

$$p = \text{const} \;\; \text{on} \;\; [0, b], \quad q \equiv 1 \;\; \text{on} \;\; [0, h];$$

hence

$$\boldsymbol{s}^{\lambda}(\boldsymbol{r}) = \begin{cases} p\boldsymbol{j} \; \text{on} \;\; (0, b) \times \{0\}, \\ \lambda\boldsymbol{i} \;\; \text{on} \;\; \{0\} \times (0, h), \\ \boldsymbol{0} \;\; \text{elsewhere on} \;\; \mathcal{S}, \end{cases}$$

see Figure 2.




The panel under special load conditions.

$$\lambda_c = pb^2/h^2.$$

$$\gamma^\lambda = \{(x,y) \in \Omega : y = \sqrt{p/\lambda}\,x\}.$$

$$\mathsf{T} = \boldsymbol{T}_{\boldsymbol{r}}^\lambda \mathcal{L}^2 \llcorner \Omega + \boldsymbol{T}_{\boldsymbol{s}}^\lambda \mathcal{H}^1 \llcorner \gamma^\lambda$$

$$\boldsymbol{T}_{\boldsymbol{r}}^\lambda(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j} \otimes \boldsymbol{j} \text{ if } \boldsymbol{r} \in \Omega_+^\lambda, \\ -\lambda \boldsymbol{i} \otimes \boldsymbol{i} \text{ if } \boldsymbol{r} \in \Omega_-^\lambda. \end{cases}$$

$$\boldsymbol{T}_{\boldsymbol{s}}^\lambda(\boldsymbol{r}) = -\sqrt{p\lambda}\,\boldsymbol{r} \otimes \boldsymbol{r}/|\boldsymbol{r}|$$

for $\boldsymbol{r} \in \gamma^\lambda$

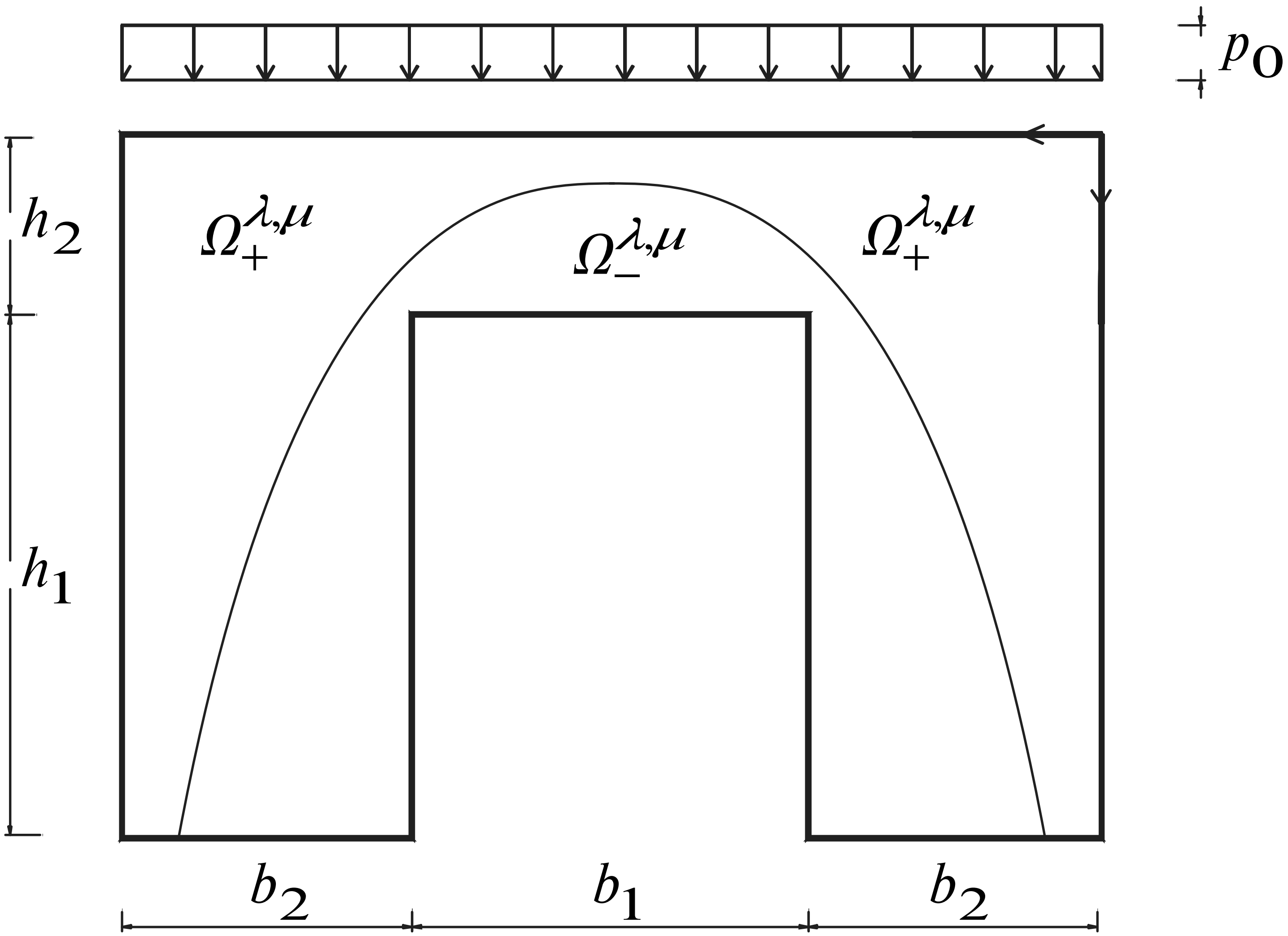
$$\boldsymbol{T}(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j} \text{ if } \boldsymbol{r}\in\Omega_+^\lambda\sim A, \\ -\mu\boldsymbol{i}\otimes\boldsymbol{i} \text{ if } \boldsymbol{r}\in\Omega_-^\lambda\sim A, \\ \boldsymbol{S}(\boldsymbol{r}) \quad \text{ if } \boldsymbol{r}\in A, \end{cases}$$

$\boldsymbol{r}\in\Omega$, where

$$\boldsymbol{S}(\boldsymbol{r}) = (2\varepsilon)^{-1}\Big((p^2x^4/y^4-(\mu+\varepsilon)^2)\boldsymbol{i}\otimes\boldsymbol{i}/2+p(\mu-\varepsilon-px^2/y^2)\boldsymbol{j}\otimes\boldsymbol{j}-2p^2x^2\boldsymbol{r}\otimes\boldsymbol{r}/y^4\Big)$$

$$\boldsymbol{T}\boldsymbol{n} = \boldsymbol{s}^\mu \quad \text{on } \mathcal{S}, \qquad \operatorname{div}\boldsymbol{T} = \boldsymbol{0} \quad \text{in } \Omega,$$

# A panel with a symmetric opening



Let $\lambda > 0, \mu > 0$ and consider the parabola

$$\gamma^{\lambda,\mu} = \{(x, \omega^{\lambda,\mu}(x)) \in \mathbf{R}^2 : b/2 - \mu < x < b/2 + \mu\}$$

where

$$\omega^{\lambda,\mu}(x) = \lambda + (h - \lambda)(x - b/2)^2/\mu^2,$$

$b/2 - \mu < x < b/2 + \mu$. Let

$$\mathcal{A} = \{(\lambda, \mu) \in (0, \infty) \times (0, \infty) : \gamma^{\lambda,\mu} \subset \Omega\}$$

$$\mathcal{A} \text{ is nonempty} \iff \zeta \le 4\xi(\xi + 1),$$

$$\mathcal{A} \text{ has a nonempty interior} \iff \zeta < 4\xi(\xi + 1)$$

where

$$\xi := b_2/b_1, \quad \zeta := h_1/h_2.$$

**Proposition.** *Let* $(\lambda,\mu)\in\mathcal{A}$ *and define the measure* $\mathsf{T}^{\lambda,\mu}$ *by*

$$\mathsf{T}^{\lambda,\mu}=\boldsymbol{T}_r^{\lambda,\mu}\mathcal{L}^2\llcorner\Omega+\boldsymbol{T}_s^{\lambda,\mu}\mathcal{H}^1\llcorner\gamma^{\lambda,\mu}$$

$$\boldsymbol{T}_r^{\lambda,\mu}(\boldsymbol{r})=\begin{cases}-p_0\boldsymbol{j}\otimes\boldsymbol{j}\ \text{if}\ \ \boldsymbol{r}\in\Omega_+^{\lambda,\mu},\\ \boldsymbol{0}\qquad\qquad\quad\text{otherwise},\end{cases}$$

$\boldsymbol{r}\in\Omega$, *and*

$$\boldsymbol{T}_s^{\lambda,\mu}(\boldsymbol{r})=\sigma^{\lambda,\mu}(\boldsymbol{r})\boldsymbol{t}^{\lambda,\mu}(\boldsymbol{r})\otimes\boldsymbol{t}^{\lambda,\mu}(\boldsymbol{r})$$

$$\sigma^{\lambda,\mu}(\boldsymbol{r})=-\frac{p_0\sqrt{\mu^4+4(h-\lambda)^2(x-b/2)^2}}{2(h-\lambda)},$$

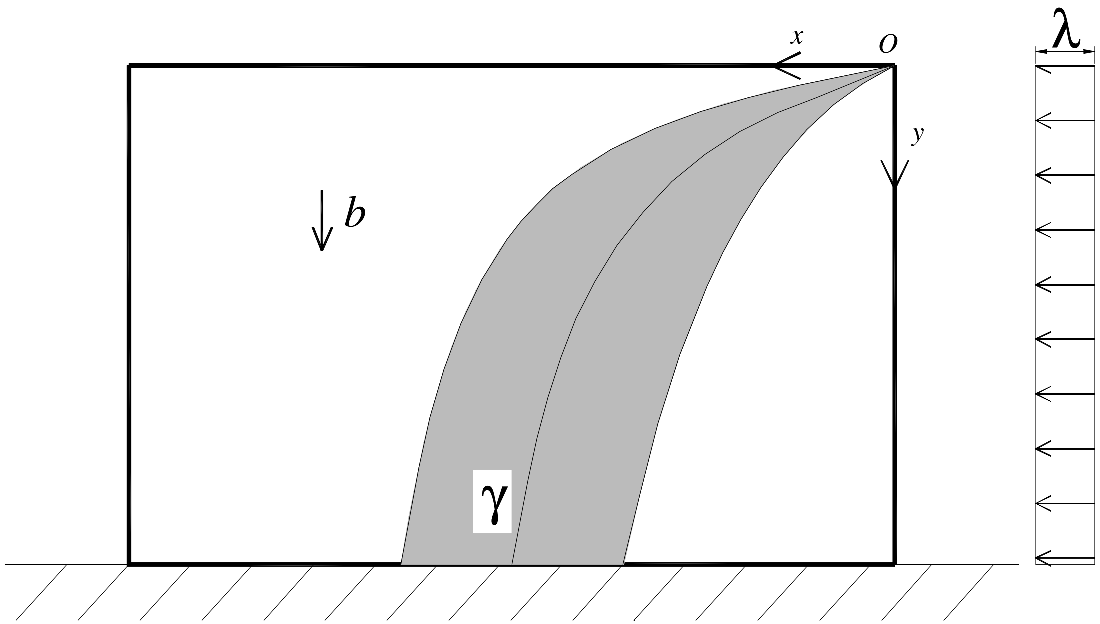
$\boldsymbol{r}=(x,y)\in\gamma^{\lambda,\mu}$. *Then* $\mathsf{T}^{\lambda,\mu}$ *is an admissible stress field weakly equilibrating the loads* $(\boldsymbol{s},\boldsymbol{0})$.

We emphasize that for all $(\lambda, \mu) \in \mathcal{A}$ the stress field $\mathrm{T}^{\lambda,\mu}$ equilibrates the same loads.

**Proposition.** *If $\mathcal{A}$ has a nonempty interior then the loads $(\boldsymbol{s}, \mathbf{0})$ are strongly compatible; in fact there exists a bounded admissible stress field $\boldsymbol{T}$ on $\Omega$ strongly equilibrating them.*

# Panel under gravity

subjected to side loads and its own gravity



*exists* $\lambda_c > 0$ *such that the loads are weakly compatible if* $0 \leq \lambda < \lambda_c$. *The corresponding stress is singular on the curve that starts at the upper right corner and ends in the base of the panel. The application of the integration shows that the loads are strongly compatible if* $0 \leq \lambda < \lambda_c$. *The averaged stress can be determined explicitly.*