From yielding function and plastic potential to pseudo-potentials and bipotential : a constructive procedure

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Constitutive laws for plasticity

$\mathbb{E} = \{\mathbf{x} \in \mathbb{V} \ / \ f(\mathbf{x}) \leq 0\}$ elastic domain
for the generalized thermodynamic forces $\mathbf{x} \in \mathbb{V}$

1. Associative flow rules
   generalized normality condition of the flow $\mathbf{x}^* \in \mathbb{V}^*$
   to the boundary of $\mathbb{E}$, (Halphen, Nguyen)
2. Non-associative flow rules
   the direction of the flow $\mathbf{x}^*$ is defined as
   the gradient of the plastic potential $g(\mathbf{x})$.

$\Longrightarrow$ introduction of pseudo-potentials to make sure that the mechanical dissipation is positive (Moreau).

Associative flow rules

Definition of pseudo-potentials
$\mathbf{x}$ generalized thermodynamic force , $\mathbf{x}^*$ flow.

$$\phi(\mathbf{x}) := \mathbb{I}_{\mathbb{E}}(\mathbf{x}) \quad \text{indicator function of the elastic domain} \mathbb{E}$$

$$\phi^*(\mathbf{x}^*) := \sup_{\mathbf{x} \in \mathbb{V}} (\mathbf{x} \cdot \mathbf{x}^* - \phi(\mathbf{x})) \quad \text{the Legendre-Fenchel conjugate of } \phi$$

$$\forall\ (\mathbf{x}, \mathbf{x}^*) \in \mathbb{V} \times \mathbb{V}^* \qquad \phi(\mathbf{x}) + \phi^*(\mathbf{x}^*) \geq \mathbf{x} \cdot \mathbf{x}^*$$

The equality is reached for a pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}^*)$ which verifies

$$\tilde{\mathbf{x}}^* \in \partial \phi(\tilde{\mathbf{x}}) \quad \text{and} \quad \tilde{\mathbf{x}} \in \partial \phi^*(\tilde{\mathbf{x}}^*)$$

and $(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}^*)$ are the real physical values.
For non-associative flow rule the definition of such pseudo-potentials is not straightforward.

Non-associative flow rule

Two different approaches :
1. Introduction of a bipotential (de Saxcé, Hjiaj, Bodovillé)
$b(\mathbf{x}, \mathbf{x}^*)$ depending on the dual variables,
the thermodynamic force $\mathbf{x}$ and the flow $\mathbf{x}^*$,
separately convex and such that :

$$\forall (\mathbf{x}, \mathbf{x}^*) \in \mathbb{V} \times \mathbb{V}^* \qquad b(\mathbf{x}, \mathbf{x}^*) \geq \mathbf{x} \cdot \mathbf{x}^*$$

Introduced for Drucker-Prager model and unilateral contact.
The stress-strain evolution is defined by the extremal couples of the bipotential.
In the simple case of associative flow rule :

$$\forall\ (\mathbf{x}, \mathbf{x}^*) \in \mathbb{V} \times \mathbb{V}^* \qquad b(\mathbf{x}, \mathbf{x}^*) := \phi(\mathbf{x}) + \phi^*(\mathbf{x}^*)$$

second approach :

2. Introduction of pseudo-potentials
depending on the state variables + Helmholtz free energy (Ziegler)

depending on generalized stresses + Gibbs free energy (Collins, Houlsby, Puzrin , etc)

several exemples : Drucker-Prager, non-linear kinematic hardening, endochronic theory, Mroz (Erlicher, Point)

Link between these two different approaches ?

Bipotential
and/or
conjugated pseudo-potentials

General theoretical framework and notations

Assumption : isothermal and infinitesimal transformations.

$\mathbf{f}^{nd} \in \mathbb{V}$ non-dissipative (quasi-conservative) thermodynamic forces defined as the gradient of the Helmholtz free density of energy $\psi(\mathbf{v})$

$$\boxed{\mathbf{f}^{nd} = \nabla\psi(\mathbf{v})} \tag{1}$$

Φ_m mechanical dissipation $\Phi_m(t) := \mathbf{f} : \dot{\mathbf{v}} - \dot{\psi} = \mathbf{f} : \dot{\mathbf{v}} - \mathbf{f}^{nd} \cdot \dot{\mathbf{v}}$

$\mathbf{f}$ external forces, $\mathbf{f}^{d}$ dissipative thermodynamic forces defined by :

$$\mathbf{f}^{d} := \mathbf{f} - \mathbf{f}^{nd}$$

2nd principle : the mechanical dissipation is assumed nonnegative :

$$\boxed{\Phi_m(t) = (\mathbf{f} - \mathbf{f}^{nd}) \cdot \dot{\mathbf{v}} = \mathbf{f}^{d} \cdot \dot{\mathbf{v}} \geq 0} \tag{2}$$

To define conjugate pseudo-potentials is a usual way to obtain (2) :

$$\boxed{\mathbf{f}^{d} \in \partial\phi^{*}(\dot{\mathbf{v}}) \qquad \text{and} \qquad \dot{\mathbf{v}} \in \partial\phi\left(\mathbf{f}^{d}\right)}$$

For the state variables $(\varepsilon, \mathbf{u}) = (\varepsilon, \varepsilon^p, \alpha)$ we use the distinction between

$$\text{dissipative forces}: \mathbf{f}^d = (\sigma^d, \mathbf{p}^d) \qquad = \left(\sigma^d, \tau^d, \mathbf{X}^d\right)$$

$$\text{non-dissipative forces}: \mathbf{f}^{nd} = (\sigma^{nd}, \mathbf{p}^{nd}) \qquad = \left(\sigma^{nd}, \tau^{nd}, \mathbf{X}^{nd}\right)$$

σ^d and σ^{nd} are the thermodynamic forces associated with ε.
τ^d and τ^{nd} are the thermodynamic forces associated with ε^p.
If the dissipation is independent of $\dot{\varepsilon}$ the corresponding actual value of the dissipative force must be 0 : $\sigma^d = 0$
hence $\sigma = \sigma^{nd}$, often σ is written instead of σ^{nd}.
The external forces are $\mathbf{f} = (\sigma, \mathbf{0}) = (\sigma, \mathbf{0}, \mathbf{0})$
Since $\mathbf{f}^d := \mathbf{f} - \mathbf{f}^{nd}$

$$\sigma^d := \sigma - \sigma^{nd} = \mathbf{0}$$
$$\mathbf{p}^d := \mathbf{0} - \mathbf{p}^{nd}$$

If moreover the Helmholtz energy depends on ε and ε^p only by

$$\frac{1}{2}\left(\varepsilon - \varepsilon^p\right) : \mathbf{C} : \left(\varepsilon - \varepsilon^p\right)$$

then $-\tau^{nd} = \mathbf{C} : (\varepsilon - \varepsilon^p) = \sigma^{nd}$ hence

$$\tau^d = -\tau^{nd} = \sigma^{nd} = \sigma$$

In classical textbooks f and g are functions of $(\sigma, \mathbf{X})$
Actually the above relationships show that f and g are functions of the dissipative forces.

$$\boxed{f = f\left(\tau^d, \mathbf{X}^d\right) = f\left(\sigma, \mathbf{X}\right), \quad g = g\left(\tau^d, \mathbf{X}^d\right) = g\left(\sigma, \mathbf{X}\right)}$$

Construction of a loading function

A plasticity model is defined by :
$\mathbb{E} = \{\mathbf{x} \in \mathbb{V} \ / \ f(\mathbf{x}) \leq 0\}$ elastic domain for thermodynamic forces
$\nabla g(\mathbf{x})$ gives the direction of the flow $\mathbf{x}^*$.
We define a loading function F as :

$$\forall(\mathbf{x},\mathbf{y}) \in \mathbb{V} \times \mathbb{V} \qquad \boxed{F(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \nabla(g-f)(\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y})}$$

$$\nabla_{\mathbf{x}} F(\mathbf{x},\mathbf{y}) = \nabla f(\mathbf{x}) + \nabla(g-f)(\mathbf{y})$$

Hence for $\mathbf{x} = \mathbf{y}$ and for any $\mathbf{y}$:

$$\boxed{F(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{y}} = f(\mathbf{y})}$$

$$\boxed{\nabla_{\mathbf{x}} F(\mathbf{x},\mathbf{y})|_{\mathbf{x}=\mathbf{y}} = \nabla g(\mathbf{y})}$$

Remark 1 : If $\mathbf{x} = \mathbf{p}^{d'}$ and $\mathbf{y} = -\mathbf{p}^{nd'}$ then $\mathbf{p}^{d'} + \mathbf{p}^{nd'} = \mathbf{x} - \mathbf{y} = 0$
Remark 2 : For any $\mathbf{y}$ the function
$F(\mathbf{x},\mathbf{y}) := f(\mathbf{x}) + \nabla(g-f)(\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y})$ is the sum of $f(\mathbf{x})$ and of a linear function of $\mathbf{x}$ hence F is convex in $\mathbf{x}$.

Construction of pseudo-potentials

$$\mathbb{E}_{\mathbf{y}} = \{\mathbf{x} \in \mathbb{V} \; / \quad F(\mathbf{x},\mathbf{y}) \leq 0\} \qquad \text{convex}$$

$$\phi\,(\mathbf{x},\mathbf{y}) := \mathbb{I}_{\mathbb{E}_{\mathbf{y}}}(\mathbf{x}) \quad \text{the indicator function of } \mathbf{E}_{\mathbf{y}}$$

$$\phi^*(\mathbf{x}^*,\mathbf{y}) := \sup_{\mathbf{x}\in\mathbb{V}}(\mathbf{x}\cdot\mathbf{x}^* - \phi(\mathbf{x},\mathbf{y})) \quad \text{the Legendre-Fenchel conjugate of } \phi$$

For any $\mathbf{y}$, $\phi^*(\mathbf{x}^*,\mathbf{y})$ is conjugate of an indicator function of a convex domain and it is also a convex function and for any $\mathbf{y}$ it is positively homogenous with respect to $\mathbf{x}^*$ and

$$\forall\;(\mathbf{x},\mathbf{y},\mathbf{x}^*) \in \mathbb{V}\times\mathbb{V}\times\mathbb{V}^* \qquad \phi(\mathbf{x},\mathbf{y}) + \phi^*(\mathbf{x}^*,\mathbf{y}) \geq \mathbf{x}\cdot\mathbf{x}^*$$

$$B(\mathbf{x},\mathbf{y},\mathbf{x}^*) := \phi(\mathbf{x},\mathbf{y}) + \phi^*(\mathbf{x}^*,\mathbf{y})$$

The function B is bi-convex in $\mathbf{x}$ and in $\mathbf{x}^*$ and such that

$$\forall\;(\mathbf{x},\mathbf{y},\mathbf{x}^*) \in \mathbb{V}\times\mathbb{V}\times\mathbb{V}^* \qquad B(\mathbf{x},\mathbf{y},\mathbf{x}^*) \geq \mathbf{x}\cdot\mathbf{x}^*$$

Construction of a bipotentiel

The dissipative forces associated to internal variables are opposite to the corresponding non-dissipative forces

$$\boxed{\mathbf{p}^{d'} = -\mathbf{p}^{nd'} \Leftrightarrow \mathbf{x} = \mathbf{y}}$$

Definition of a bipotential

For $\mathbf{x} = \mathbf{y}$,B(x,x,x*)$define abi - function$b(x,x*) :

$$\boxed{\forall\ (\mathbf{x}, \mathbf{x}^*) \in \mathbb{V} \times \mathbb{V}^* \qquad b(\mathbf{x}, \mathbf{x}^*) := \phi(\mathbf{x}, \mathbf{x}) + \phi^*(\mathbf{x}^*, \mathbf{x})}$$

The function b is bi-convex in $\mathbf{x}$ and in $\mathbf{x}^*$ and such that

$$\forall\ (\mathbf{x}, \mathbf{x}^*) \in \mathbb{V} \times \mathbb{V}^* \qquad b(\mathbf{x}, \mathbf{x}^*) \geq \mathbf{x} \cdot \mathbf{x}^*$$

So there exists a couple $(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}^*)$ such that $b(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}^*) = \tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}}^*$

Drucker-Prager model
Non-associative elasto-perfectly plastic model :
$(\varepsilon, \mathbf{u}) = (\varepsilon, \varepsilon^p)$ state variables
$(\sigma^d, \mathbf{p}^d) = (\sigma^d, \tau^d)$ thermodynamic forces

$$f(\sigma) = \frac{\|dev(\sigma)\|}{k_d} + \frac{tr(\sigma)}{3}\tan\varphi - c$$

$$tr(\dot{\varepsilon}^p) = \dot{\lambda}\tan\theta \qquad \text{with} \quad 0 < \tan\theta < \tan\varphi$$

$$dev(\dot{\varepsilon}^p) = \dot{\lambda}\frac{1}{k_d}\frac{dev(\sigma)}{\|dev(\sigma)\|}$$

$$\dot{\lambda} f(\sigma) = 0, \quad \dot{\lambda} \geq 0, \quad f(\sigma) \leq 0$$

$\mathbf{p}^d = (tr(\tau^d), dev(\tau^d)),$

$$f\left(\mathbf{p}^d\right) = \frac{\left\|dev\left(\tau^d\right)\right\|}{k_d} + \frac{tr\left(\tau^d\right)}{3}\tan\varphi - c$$
$$g\left(\mathbf{p}^d\right) = \frac{\left\|dev\left(\tau^d\right)\right\|}{k_d} + \frac{tr\left(\tau^d\right)}{3}\tan\theta$$
$$\nabla\left(g - f\right)\left(\mathbf{p}^d\right) = \left(\tan\theta - \tan\varphi; 0\right) \quad \text{constant}$$

Loading function F

$$F\left(\mathbf{p}^{d'}, \mathbf{y}\right) = f\left(\mathbf{p}^{d'}\right) + \left(\tan\theta - \tan\varphi\right)\left(tr(\mathbf{p}^{d'} - \mathbf{y})\right)$$
$$\dot{\mathbf{u}}_y \in \partial_{\mathbf{p}^d}\mathbb{I}_{\mathbb{E}_\mathbf{y}}(\mathbf{p}^{d'})$$
$$\text{for} \quad \mathbf{p}^{d'} = \tau^{d'} = \mathbf{y}$$
$$tr\left(\dot{\varepsilon}^p\right) = \dot{\lambda}(\tan\varphi + (\tan\theta - \tan\varphi))$$
$$dev\left(\dot{\varepsilon}^p\right) = \dot{\lambda}\frac{1}{k_d}\frac{dev\left(\tau^d\right)}{\|dev(\tau^d)\|}$$
$$\dot{\lambda} f\left(\mathbf{p}^d\right) \leq 0 = 0, \quad \dot{\lambda} \geq 0, \quad f\left(\mathbf{p}^d\right) \leq 0$$

Pseudo-potentials

$$\phi\left(\tau^{d}, \mathbf{y}\right) = \mathbb{I}_{F(\tau^{d},\mathbf{y})\leq 0}\left(\tau^{d}\right)$$
$$\phi^{*}(\dot{\varepsilon}^{p}, \mathbf{y}) = \left(c - \frac{tr(\mathbf{y})}{3}\tan\varphi\right) k_{d}\left\|dev\left(\dot{\varepsilon}^{p}\right)\right\| + \mathbb{I}_{k_{d}\tan\theta\|dev(\dot{\varepsilon}^{p})\|\leq tr(\dot{\varepsilon}^{p})}$$

Bipotential

$$b\left(\dot{\varepsilon}^{p}, \tau^{d}\right) := \phi^{*}\left(\dot{\varepsilon}^{p}; \tau^{d}\right) + \phi\left(\tau^{d}; \tau^{d}\right)$$
$$b = \left(c - \frac{tr\left(\tau^{d}\right)}{3}\left(\tan\varphi - \tan\theta\right)\right)\frac{1}{\tan\theta} tr\left(\dot{\varepsilon}^{p}\right)$$
$$+\mathbb{I}_{k_{d}\tan\theta\|dev(\dot{\varepsilon}^{p})\|\leq tr(\dot{\varepsilon}^{p})} + \mathbb{I}_{f(\tau^{d})\leq 0}$$

Non-linear Kinematic Hardening :classical formulation

$$\sigma = \mathbf{C} : (\varepsilon - \varepsilon^p), \qquad \mathbf{X}^d = \mathbf{D} : (\varepsilon^p - \beta)$$
$$\dot{\varepsilon}^p = \dot{\lambda} \frac{dev\left(\sigma - \mathbf{X}^d\right)}{\left\| dev\left(\sigma - \mathbf{X}^d\right)\right\|}, \qquad \dot{\beta} = \dot{\lambda} k \mathbf{X}^d$$
$$f\left(\sigma - \mathbf{X}^d\right) = \left\| dev\left(\sigma - \mathbf{X}^d\right)\right\| - \sqrt{\frac{2}{3}}\sigma_y,$$
$$\dot{\lambda} f = 0, \;\; \dot{\lambda} \geq 0, \;\; f \leq 0$$
$$\frac{df}{dt} = 0 \quad \longrightarrow \dot{\lambda} = \frac{\mathbf{n}:\mathbf{C}:\dot{\varepsilon}}{\mathbf{n}:\mathbf{C}:\mathbf{n} + \mathbf{n}:\mathbf{D}:\mathbf{n} - k\ \mathbf{n}:\mathbf{D}:\mathbf{X}^d}$$

Often $\alpha = \varepsilon^p - \beta$ is used instead of β (Chaboche).
The force $\mathbf{X}^d$ is associated with the kinematic hardening.
Our formulation

$$f\left(\tau^d, \mathbf{X}^d\right) = \left\| dev\left(\tau^d\right)\right\| - \tilde{\sigma}_y \quad \text{with} \quad \tilde{\sigma}_y = \sqrt{\frac{2}{3}}\sigma_y$$

An additional scalar variable ζ can be added.
$\mathbf{v} = (\varepsilon, \mathbf{u}) = (\varepsilon, \varepsilon^p, \beta, \zeta)$, state variables

$$\Psi = \frac{1}{2}(\varepsilon - \varepsilon^p) : \mathbf{C} : (\varepsilon - \varepsilon^p) + \frac{1}{2}(\varepsilon^p - \beta) : \mathbf{D} : (\varepsilon^p - \beta)$$

non-dissipative forces :

$$\boxed{\begin{array}{l} \sigma^{nd} = \mathbf{C} : (\varepsilon - \varepsilon^p) \\ \tau^{nd} = -\mathbf{C} : (\varepsilon - \varepsilon^p) + \mathbf{D} : (\varepsilon^p - \beta) \\ \mathbf{X}^{nd} = -\mathbf{D} : (\varepsilon^p - \beta) \\ R^{nd} = 0 \\ \mathbf{p}^{nd} = \left(\tau^{nd}, \mathbf{X}^{nd}, R^{nd}\right) \end{array}}$$

with $\left(\tau^{d'}, \mathbf{X}^{d'}, R^{d'}\right) \in \mathbb{S}_d^2 \times \mathbb{S}_d^2 \times \mathbb{R}$:

$$\boxed{\begin{array}{c} \tilde{f}\left(\tau^{d'}, \mathbf{X}^{d'}, R^{d'}\right) = f\left(\tau^{d'}, \mathbf{X}^{d'}\right) + R^{d'} \\ = \left\|\tau^{d'}\right\| - \tilde{\sigma}_y + R^{d'} \\ \tilde{g} = \left\|\tau^{d\prime}\right\| - \tilde{\sigma}_y + \frac{1}{2}k\left\|\mathbf{X}^{d'}\right\|^2 + R^{d'} \end{array}}$$

Hence

$$\nabla(\tilde{g}-\tilde{f})\left(\tau^{d'},\mathbf{X}^{d'},R^{d'}\right)=\begin{pmatrix}0\\ k\mathbf{X}^{d'}\\ 0\end{pmatrix}$$

Loading function F

$$F(\mathbf{x},\mathbf{y})=\tilde{f}(\mathbf{x})+\nabla(\tilde{g}-\tilde{f})(\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})$$
$$\text{with}\quad \mathbf{x}=\mathbf{p}^{d'}\quad\text{and}\quad \mathbf{y}=-\mathbf{p}^{nd'}$$
$$F(\mathbf{p}^{d'},-\mathbf{p}^{nd'})=\tilde{f}\left(\tau^{d'},\mathbf{X}^{d'},R^{d'}\right)-k.\mathbf{X}^{nd'}\cdot(\mathbf{X}^{d'}+\mathbf{X}^{nd'})$$

Corresponding flow rule

$$\dot{\varepsilon}^{p}=\dot{\lambda}\frac{1}{k}\frac{\tau^{d}}{\|\tau^{d}\|}$$
$$\dot{\beta}=\dot{\lambda}k(-\mathbf{X}^{nd})=\dot{\lambda}k(-\mathbf{X}^{d})$$
$$\dot{\zeta}=\dot{\lambda}$$
$$\dot{\lambda}\tilde{f}\left(\tau^{d},\mathbf{X}^{d},R^{d}\right)\leq 0=0,\quad \dot{\lambda}\geq 0,\quad \tilde{f}\left(\tau^{d},\mathbf{X}^{d},R^{d}\right)\leq 0$$

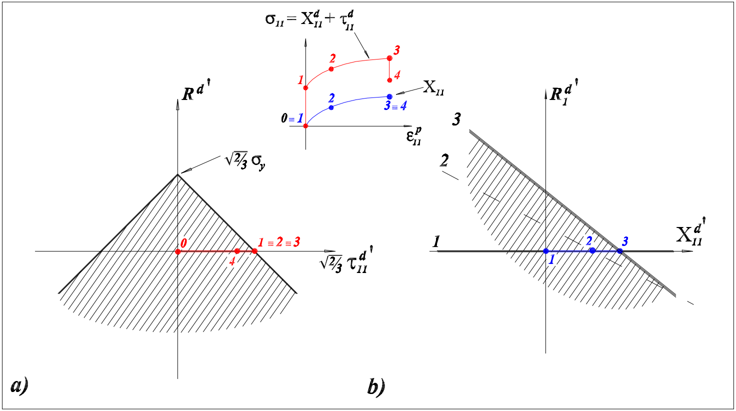

FIG.: Non-linear Kinematic Hardening .Variation of the elastic domain $\mathbb{E}_{\mathbf{y}}$.

For endochronic theory, no elastic domain.

$$\sigma = \mathbf{C} : (\varepsilon - \varepsilon^p) , \qquad \dot{\varepsilon}^p = \tilde{\beta}\ dev(\sigma)\dot{\zeta}, \qquad \dot{\zeta} \geq 0 \quad \text{with } \tilde{\beta} = \frac{\beta}{2G}$$

$\mathbf{v} = (\varepsilon, \mathbf{u}) = (\varepsilon, \varepsilon^p, \zeta)$ state variables, with ζ the intrinsic time .

$$\Psi = \frac{1}{2} (\varepsilon - \varepsilon^p) : \mathbf{C} : (\varepsilon - \varepsilon^p)$$

non-dissipative forces in $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}$

$$\begin{aligned} \sigma^{nd} &= \mathbf{C} : (\varepsilon - \varepsilon^p) \\ \mathbf{p}^{nd} &= \left(\tau^{nd}, R^{nd}\right) = \left(-\mathbf{C} : (\varepsilon - \varepsilon^p) , 0\right) \end{aligned}$$

dissipative forces in $\mathbb{S}^2_d \times \mathbb{R}$

$$\boxed{\begin{aligned} f(\tau^d, R^d) &= R^d \\ \nabla g(\tau^d, R^d) &= \begin{pmatrix} \tilde{\beta}\ dev(\tau^d) \\ 1 \end{pmatrix} \end{aligned}}$$

$$F\left(\mathbf{x},\mathbf{y}\right)=\tilde{f}(\mathbf{x})+\nabla(\tilde{g}-\tilde{f})(\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})$$
$$\text{with}\quad \mathbf{x}=\mathbf{p}^{d'}\quad\text{and}\quad \mathbf{y}=-\mathbf{p}^{nd'}=-\left(\tau^{nd'},R^{nd\prime}\right)\in\mathbb{S}_d^2\times\mathbb{R}$$
$$F\left(\mathbf{p}^{d'},-\mathbf{p}^{nd'}\right)=\tilde{\beta}\left(\tau^{d'}:\left(-\tau^{nd'}\right)-\left\|\tau^{nd'}\right\|^2\right)+R^{d'}\leq 0\}$$

Hence

$$\dot{\varepsilon}^p=\dot{\lambda}\,\tilde{\beta}\left(-dev(\tau^{nd'})\right)\Big|_{\mathbf{y}=\mathbf{p}^d}=\dot{\lambda}\tilde{\beta}\;dev\left(\tau^d\right)$$
$$\dot{\zeta}=\dot{\lambda}$$
$$\text{with}\quad \mathbf{y}=-\mathbf{p}^{nd'}=\mathbf{p}^{d'}\qquad F=0\ \text{hence}$$
$$\dot{\lambda}F\leq 0\quad\text{and}\quad F\leq 0\quad \dot{\lambda}\geq 0$$

only condition : non-negativity of $\dot{\lambda}$ the intrinsic time increment.

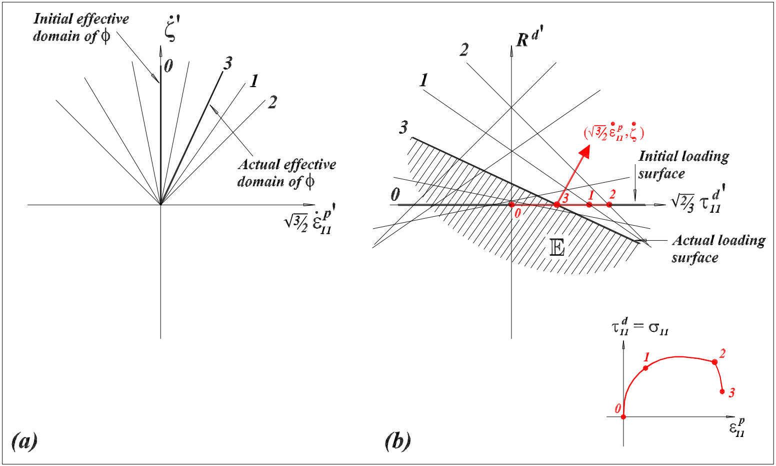

FIG.: Endochronic theory .

a) Variation of $\mathbb{D}_{\mathbf{y}}$, domain of pseudo-potentiel . b) Variation of $\mathbb{E}_{\mathbf{y}}$.

$\mathbf{u}$ displacement of a point M with respect to a fixed plan rigid body.
($u_n < 0$ impossible, $u_n = 0$ contact, $u_n > 0$ no contact)
$\dot{\mathbf{u}} = \dot{\mathbf{u}}_t + \dot{u}_n \ \mathbf{n}$ velocity
$\mathbf{r}^d = \mathbf{r}_t^d + r_n^d \ \mathbf{n}$ dissipative reaction of the rigid plan
it must remain in the convex $K = \{r \ / \quad \|\mathbf{r}_t\| \leq \mu r_n\}$
Description of the model

$$f(\mathbf{r}^d) := \left\|\mathbf{r}_t^d\right\| - \mu r_n^d$$

$$\dot{\mathbf{u}} = \dot{\lambda} \nabla g(\mathbf{r}^d) \iff \begin{cases} \dot{\mathbf{u}}_t = \dot{\lambda} \frac{\mathbf{r}_t^d}{\|\mathbf{r}_t^d\|} \\ \dot{u}_n = 0 \end{cases}$$

$$\text{since} \quad \nabla f(\mathbf{r}^d) = \begin{cases} \frac{\mathbf{r}_t^d}{\|\mathbf{r}_t^d\|} \\ -\mu \end{cases} \quad \text{hence} \quad \nabla(g - f)(\mathbf{r}^d) = \begin{cases} 0 \\ \mu \end{cases}$$

Loading function F

$$
\begin{aligned}
F\left(\mathbf{r}^{d'}, \mathbf{y}\right) &= f\left(\mathbf{r}^{d'}\right) + \nabla\left(g - f\right)(\mathbf{y}) \cdot \left(\mathbf{r}^{d'} - \mathbf{y}\right) \\
&= \left\|\mathbf{r}_t^{d'}\right\| - \mu r_n^{d'} + \mu(r_n^{d'} - y_n) \\
&= \left\|\mathbf{r}_t^{d'}\right\| - \mu y_n
\end{aligned}
$$

Since $\mathbf{y}$ should be replaced by $\mathbf{r}^{d'}$ and since $r_n^{d'} \geq 0$ then $y_n \geq 0$

Pseudo- potentials

$$
\begin{aligned}
\phi(\mathbf{r}^{d'}, \mathbf{y}) &= \mathbb{I}_{F\leq 0}(\mathbf{r}^{d'}) \\
\phi^*(\dot{\mathbf{u}}', \mathbf{y}) &= \mathbb{I}_{R^-}(\dot{u}'_n) + \mu y_n \left\|\dot{\mathbf{u}}'_t\right\|
\end{aligned}
$$

For $\mathbf{y} = \mathbf{x} = \mathbf{r}^{d'}$ de Saxcé bipotential

$$
\begin{aligned}
b(\dot{\mathbf{u}}', \mathbf{r}^{d'}) &= \phi^*(\dot{\mathbf{u}}', \mathbf{r}^{d'}) + \phi(\mathbf{r}^{d'}, \mathbf{r}^{d'}) \\
&= \mathbb{I}_{R^-}(\dot{u}'_n) + \mu r_n^{d'} \left\|\dot{\mathbf{u}}'_t\right\| + \mathbb{I}_K(\mathbf{r}^{d'})
\end{aligned}
$$

Constructive procedure

Our constructive procedure permits to construct a loading function and then the corresponding pseudo-potentials and a bipotential.
Based on the distinction between dissipative and non dissipative thermodynamic forces and on their relationship.
The method is illustrated by four examples :

1. non-associative Drucker-Prager model,
2. non-linear kinematic model,
3. Bouc-Wen endochronic theory,
4. unilateral-contact with dry friction,

The same sort of procedure can be used to define other rate-independent flow rules

Thank you for your attention
Any questions ?