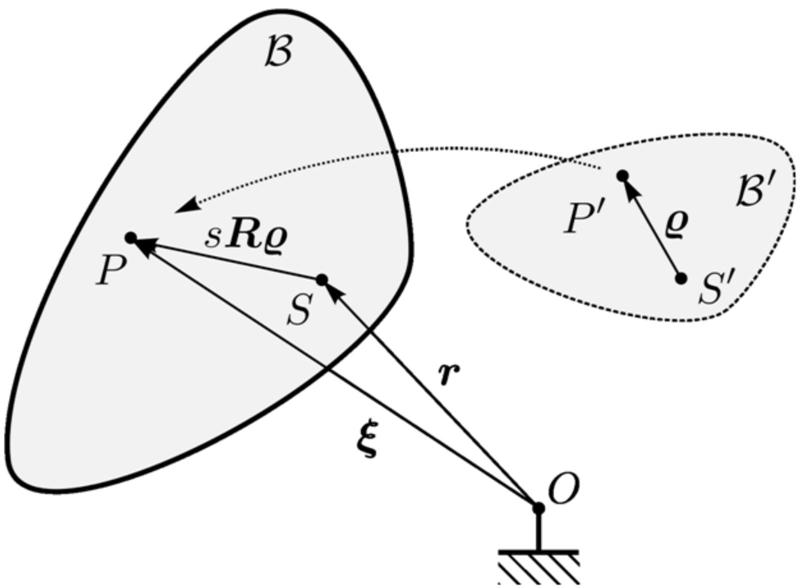
Rigid Body Dynamics with Quaternions and Perfect Constraints

Michael Möller and Christoph Glocker



$$\begin{cases} m\dot{\boldsymbol{v}} = \boldsymbol{F} \\ \frac{1}{2}\operatorname{Tr}\boldsymbol{\Theta}\dot{\nu} - \frac{1}{|A|^2}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{\Theta}\boldsymbol{\omega} = c_S \\ \boldsymbol{\Theta}\dot{\boldsymbol{\omega}} + \frac{1}{|A|^2}(\nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}})\boldsymbol{\Theta}\boldsymbol{\omega} = \boldsymbol{M}_S \end{cases}$$

$$\begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v} \\ \dot{A} = \frac{1}{2|A|^2}A(\nu, \boldsymbol{\omega}) \end{cases}$$

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Center of Mechanics
Prof. Dr.-Ing. Ch. Glocker
ETH Zentrum
CH- 8092 Zürich

IMES
Institute of Mechanical Systems

Quaternions and Rotations

- Quaternion $A = (a_0, \boldsymbol{a}) = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}, \quad a_i \in \mathbb{R}$

- Conjugation and norm $\tilde{A} = (a_0, -\boldsymbol{a}) \qquad |A| = \sqrt{a_0^2 + \boldsymbol{a}^\top \boldsymbol{a}}$

- Multiplication

$$AB = (a_0 b_0 - \boldsymbol{a}^\top \boldsymbol{b}, a_0 \boldsymbol{b} + b_0 \boldsymbol{a} + \tilde{\boldsymbol{a}} \boldsymbol{b}) \qquad B = (b_0, \boldsymbol{b}) \in \mathbb{H} \qquad AB \neq BA$$

not commutative

- Rotation and scaling of vectors

$$A(0, \boldsymbol{x})\tilde{A} = (0, s\boldsymbol{R}\boldsymbol{x}), \quad \boldsymbol{R} \in \mathrm{SO}(3), \quad s \in \mathbb{R}_0^+$$

$$\boldsymbol{R} := \frac{1}{|A|^2} \begin{pmatrix} \boldsymbol{a} & a_0 \boldsymbol{I} + \tilde{\boldsymbol{a}} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}^\top \\ a_0 \boldsymbol{I} + \tilde{\boldsymbol{a}} \end{pmatrix}, \quad s := |A|^2$$

Kinematics

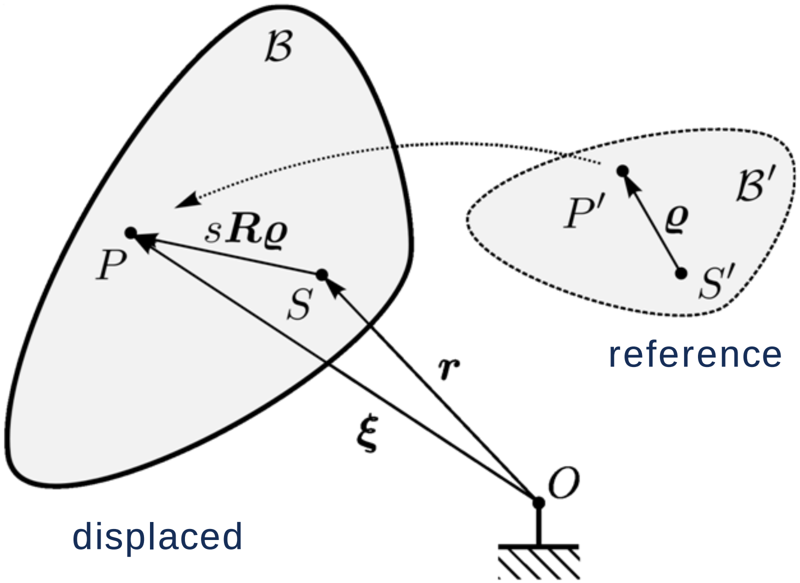
- Body with 3 translational, 3 rotational and 1 scaling degree of freedom

$$\boldsymbol{\xi} = s\boldsymbol{R}\boldsymbol{\varrho} + \boldsymbol{r}$$ 7 degrees of freedom

$\boldsymbol{\xi}$: actual position
S : reference position

$\boldsymbol{R}$: rotation matrix
s : scaling factor
$\boldsymbol{r}$: translation vector

$\left.\begin{matrix}\boldsymbol{R}\\ s\end{matrix}\right\} A = (a_0, \boldsymbol{a})$



- Generalized coordinates

$$\boldsymbol{q} := \begin{pmatrix} \boldsymbol{r} \\ a_0 \\ \boldsymbol{a} \end{pmatrix}, \quad |A| \neq 0$$

- Kinematics

$$(0, \boldsymbol{\xi}) = A(0, \boldsymbol{\varrho})\tilde{A} + (0, \boldsymbol{r})$$ absolute position

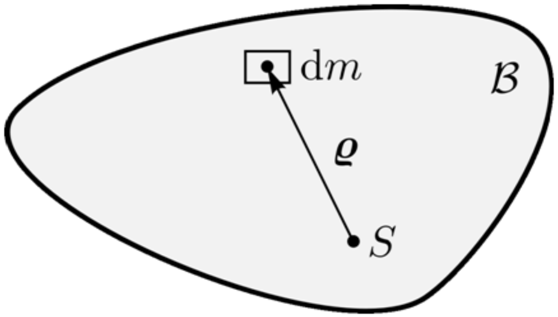
$$\dot{\boldsymbol{\xi}} = (\boldsymbol{I} \quad \boldsymbol{R}\boldsymbol{\varrho} \quad -\boldsymbol{R}\tilde{\boldsymbol{\varrho}})\boldsymbol{Q}\dot{\boldsymbol{q}}$$ absolute velocity

$$\boldsymbol{Q} := \begin{pmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & 2a_0 & 2\boldsymbol{a}^\mathsf{T} \\ 0 & -2\boldsymbol{a} & 2a_0\boldsymbol{I} - 2\tilde{\boldsymbol{a}} \end{pmatrix}$$

Kinetic Energy

- Kinematics $\dot{\boldsymbol{\xi}} = (\boldsymbol{I} \quad \boldsymbol{R}\boldsymbol{\varrho} \quad -\boldsymbol{R}\tilde{\boldsymbol{\varrho}})\boldsymbol{Q}\dot{\boldsymbol{q}}$

reference configuration



S : center of mass

- Kinetic energy

$$T = \frac{1}{2}\int_{\mathcal{B}} \dot{\boldsymbol{\xi}}^{\mathsf{T}}\dot{\boldsymbol{\xi}}\,\mathrm{d}m$$

$$= \frac{1}{2}\dot{\boldsymbol{q}}^{\mathsf{T}}\boldsymbol{Q}^{\mathsf{T}}\int_{\mathcal{B}}\begin{pmatrix} \boldsymbol{I} & \boldsymbol{R}\boldsymbol{\varrho} & -\boldsymbol{R}\tilde{\boldsymbol{\varrho}} \\ \boldsymbol{\varrho}^{\mathsf{T}}\boldsymbol{R}^{\mathsf{T}} & \boldsymbol{\varrho}^{\mathsf{T}}\boldsymbol{\varrho} & 0 \\ \tilde{\boldsymbol{\varrho}}\boldsymbol{R}^{\mathsf{T}} & 0 & -\tilde{\boldsymbol{\varrho}}\tilde{\boldsymbol{\varrho}} \end{pmatrix}\mathrm{d}m\,\boldsymbol{Q}\dot{\boldsymbol{q}}$$

- Mass and inertia tensor

$$m := \int_{\mathcal{B}}\mathrm{d}m, \quad \boldsymbol{\Theta} := \int_{\mathcal{B}}\tilde{\boldsymbol{\varrho}}\tilde{\boldsymbol{\varrho}}^{\mathsf{T}}\mathrm{d}m$$

$$\int_{\mathcal{B}}\boldsymbol{\varrho}\,\mathrm{d}m = 0, \quad \int_{\mathcal{B}}\boldsymbol{\varrho}^{\mathsf{T}}\boldsymbol{\varrho}\,\mathrm{d}m = \frac{1}{2}\mathrm{Tr}\,\boldsymbol{\Theta}$$

- Mass matrix

$$\boldsymbol{M} := \begin{pmatrix} m\boldsymbol{I} & 0 & 0 \\ 0 & \frac{1}{2}\mathrm{Tr}\,\boldsymbol{\Theta} & 0 \\ 0 & 0 & \boldsymbol{\Theta} \end{pmatrix} \in \mathbb{R}^{7\times 7}$$

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathsf{T}}\boldsymbol{Q}^{\mathsf{T}}\boldsymbol{M}\boldsymbol{Q}\dot{\boldsymbol{q}}$$

kinetic energy of the body

Virtual Work

- Principle of virtual work, dynamics of the infinite dimensional system

$$\delta W = \int_{\mathcal{S}} \delta\boldsymbol{\xi}^{\mathsf{T}}(\ddot{\boldsymbol{\xi}}\,\mathrm{d}m - \mathrm{d}\boldsymbol{F} - \mathrm{d}\boldsymbol{Z}) = 0, \quad \forall\,\delta\boldsymbol{\xi} \quad \Leftrightarrow \quad \text{System } \mathcal{S} \text{ is in dynamic equilibrium.}$$

- Perfect bilateral constraint $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\varrho}, \boldsymbol{q}, t), \quad \int_{\mathcal{S}} \delta\boldsymbol{\xi}^{\mathsf{T}}\mathrm{d}\boldsymbol{Z} = 0, \quad \delta\boldsymbol{\xi} = \frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\,\delta\boldsymbol{q} \quad \forall\,\delta\boldsymbol{q}$
- Reduction to system with $\boldsymbol{q}$ coordinates

$$\delta\boldsymbol{q}^{\mathsf{T}}\int_{\mathcal{S}}\left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathsf{T}}\ddot{\boldsymbol{\xi}}\,\mathrm{d}m - \delta\boldsymbol{q}^{\mathsf{T}}\underbrace{\int_{\mathcal{S}}\left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathsf{T}}\mathrm{d}\boldsymbol{F}}_{=:\boldsymbol{f}} = 0, \quad \forall\,\delta\boldsymbol{q}$$

$$\left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathsf{T}}\ddot{\boldsymbol{\xi}} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathsf{T}}\dot{\boldsymbol{\xi}}\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathsf{T}}\dot{\boldsymbol{\xi}}$$

$$\delta\boldsymbol{q}^{\mathsf{T}}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}}\left(\frac{\partial\dot{\boldsymbol{\xi}}}{\partial\dot{\boldsymbol{q}}}\right)^{\mathsf{T}}\dot{\boldsymbol{\xi}}\mathrm{d}m - \delta\boldsymbol{q}^{\mathsf{T}}\int_{\mathcal{S}}\left(\frac{\partial\dot{\boldsymbol{\xi}}}{\partial\boldsymbol{q}}\right)^{\mathsf{T}}\dot{\boldsymbol{\xi}}\mathrm{d}m - \delta\boldsymbol{q}^{\mathsf{T}}\boldsymbol{f} = 0, \quad \forall\,\delta\boldsymbol{q}$$

$$\delta\boldsymbol{q}^{\mathsf{T}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\boldsymbol{q}}}\right)^{\mathsf{T}} - \delta\boldsymbol{q}^{\mathsf{T}}\left(\frac{\partial T}{\partial\boldsymbol{q}}\right)^{\mathsf{T}} - \delta\boldsymbol{q}^{\mathsf{T}}\boldsymbol{f} = 0, \quad \forall\,\delta\boldsymbol{q} \qquad T(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) := \frac{1}{2}\int_{\mathcal{S}}\dot{\boldsymbol{\xi}}^{\mathsf{T}}\dot{\boldsymbol{\xi}}\mathrm{d}m$$

Equations of motion

Equations of Motion

- Principle of virtual work:

$$\delta \boldsymbol{q}^\mathsf{T} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\boldsymbol{q}}}\right)^\mathsf{T} - \delta \boldsymbol{q}^\mathsf{T}\left(\frac{\partial T}{\partial \boldsymbol{q}}\right)^\mathsf{T} - \delta \boldsymbol{q}^\mathsf{T}\boldsymbol{f} = 0, \quad \forall\, \delta \boldsymbol{q}$$

$$\boldsymbol{f} = \int_{\mathcal{B}} \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^\mathsf{T} \mathrm{d}\boldsymbol{F}$$

- Partial derivative

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^\mathsf{T}\boldsymbol{Q}^\mathsf{T}\boldsymbol{M}\boldsymbol{Q}\dot{\boldsymbol{q}} = \frac{m}{2}\dot{\boldsymbol{r}}^\mathsf{T}\dot{\boldsymbol{r}} + \frac{1}{2}\boldsymbol{q}^\mathsf{T}\dot{\boldsymbol{Q}}^\mathsf{T}\boldsymbol{M}\dot{\boldsymbol{Q}}\boldsymbol{q}$$

$$\left(\frac{\partial T}{\partial \dot{\boldsymbol{q}}}\right)^\mathsf{T} = \boldsymbol{Q}^\mathsf{T}\boldsymbol{M}\boldsymbol{Q}\dot{\boldsymbol{q}}, \qquad \left(\frac{\partial T}{\partial \boldsymbol{q}}\right)^\mathsf{T} = \dot{\boldsymbol{Q}}^\mathsf{T}\boldsymbol{M}\dot{\boldsymbol{Q}}\boldsymbol{q}$$

- Equations of motion in generalized coordinates

$$\boldsymbol{Q}^\mathsf{T}\boldsymbol{M}\boldsymbol{Q}\ddot{\boldsymbol{q}} + \boldsymbol{Q}^\mathsf{T}\boldsymbol{M}\dot{\boldsymbol{Q}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{Q}}^\mathsf{T}\boldsymbol{M}(\boldsymbol{Q}\dot{\boldsymbol{q}} - \dot{\boldsymbol{Q}}\boldsymbol{q}) - \boldsymbol{f} = 0$$

7 degrees of freedom

$$\boldsymbol{f} = \int_{\mathcal{B}} \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^\mathsf{T} (\mathrm{d}\boldsymbol{F}^a + \mathrm{d}\boldsymbol{F}^z), \qquad \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}} = \frac{\partial \dot{\boldsymbol{\xi}}}{\partial \dot{\boldsymbol{q}}} = \begin{pmatrix}\boldsymbol{I} & \boldsymbol{R}\boldsymbol{\varrho} & -\boldsymbol{R}\tilde{\boldsymbol{\varrho}}\end{pmatrix}\boldsymbol{Q}$$

$$\boldsymbol{F} := \int_{\mathcal{B}} \mathrm{d}\boldsymbol{F}^a, \quad c_S := \int_{\mathcal{B}} \boldsymbol{\varrho}^\mathsf{T}\boldsymbol{R}^\mathsf{T}\mathrm{d}\boldsymbol{F}^a, \quad \boldsymbol{M}_S := \int_{\mathcal{B}} \tilde{\boldsymbol{\varrho}}\boldsymbol{R}^\mathsf{T}\mathrm{d}\boldsymbol{F}^a$$

$$\boldsymbol{z} := \int_{\mathcal{B}} \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{q}}\right)^\mathsf{T} \mathrm{d}\boldsymbol{F}^z$$

$$\boldsymbol{f} = \boldsymbol{Q}^\mathsf{T}\begin{pmatrix}\boldsymbol{F}\\ c_S\\ \boldsymbol{M}_S\end{pmatrix} + \boldsymbol{z}$$

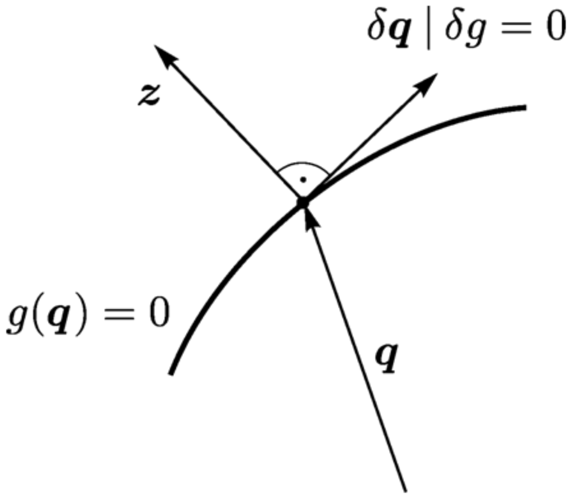
Perfect Constraining of the Scaling

- Perfect bilateral constraint (d'Alembert / Lagrange)
  - Constraint equation $g(\boldsymbol{q}) = |A|^2 - 1 = 0$
  - Constraint force $\delta\boldsymbol{q}^\mathsf{T}\boldsymbol{z} = 0 \quad \forall\delta\boldsymbol{q} \mid \delta g = 0$
- Reformulation of the constraint force

$$\delta g = \frac{\partial g}{\partial \boldsymbol{q}}\delta\boldsymbol{q}$$

$$\delta\boldsymbol{q}^\mathsf{T}\boldsymbol{z} = 0 \quad \forall\delta\boldsymbol{q} \mid \delta\boldsymbol{q}^\mathsf{T}\left(\frac{\partial g}{\partial \boldsymbol{q}}\right)^\mathsf{T} = 0$$

$$\boldsymbol{z} = \left(\frac{\partial g}{\partial \boldsymbol{q}}\right)^\mathsf{T}\lambda = \begin{pmatrix} 0 \\ 2a_0 \\ 2\boldsymbol{a} \end{pmatrix}\lambda, \quad \lambda \in \mathbb{R}$$



- Equations of motion of a rigid body (DAE)

$$\boldsymbol{Q}^\mathsf{T}\boldsymbol{M}\boldsymbol{Q}\ddot{\boldsymbol{q}} + \boldsymbol{Q}^\mathsf{T}\boldsymbol{M}\dot{\boldsymbol{Q}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{Q}}^\mathsf{T}\boldsymbol{M}(\boldsymbol{Q}\dot{\boldsymbol{q}} - \dot{\boldsymbol{Q}}\boldsymbol{q}) - \boldsymbol{Q}^\mathsf{T}(\boldsymbol{F}^\mathsf{T}, c_S + \lambda, \boldsymbol{M}_S^\mathsf{T})^\mathsf{T} = 0 \qquad |A|^2 = 1$$

Transformation of the Velocities

- New generalized velocities

$$\boldsymbol{u} := \begin{pmatrix} \boldsymbol{v} \\ \nu \\ \boldsymbol{\omega} \end{pmatrix} = \boldsymbol{Q}\dot{\boldsymbol{q}}$$

$\boldsymbol{v} = \dot{\boldsymbol{r}}$	: velocity
$\nu = \dot{s}$	: scaling velocity
$\tilde{\boldsymbol{\omega}} = s\boldsymbol{R}^{\mathsf{T}}\dot{\boldsymbol{R}}$	: generalized angular velocity

$$\dot{\boldsymbol{q}} = \boldsymbol{Q}^{-1}\boldsymbol{u}, \quad \delta\dot{\boldsymbol{q}} = \boldsymbol{Q}^{-1}\delta\boldsymbol{u}$$

$$\delta\dot{\boldsymbol{q}}^{\mathsf{T}}(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{\mathsf{T}}\boldsymbol{M}\boldsymbol{Q}\dot{\boldsymbol{q}}) - \dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}\dot{\boldsymbol{Q}}\boldsymbol{q} - \boldsymbol{f}) = 0 \quad \forall\delta\dot{\boldsymbol{q}}$$

Principle of virtual power

$$\delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{M}\dot{\boldsymbol{u}} + \delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{Q}^{-\mathsf{T}}\dot{\boldsymbol{Q}}^{\mathsf{T}}\boldsymbol{M}(\boldsymbol{u} - \dot{\boldsymbol{Q}}\boldsymbol{q}) - \delta\boldsymbol{u}^{\mathsf{T}}\boldsymbol{Q}^{-\mathsf{T}}\boldsymbol{f} = 0 \quad \forall\delta\boldsymbol{u}$$

$$\boldsymbol{Q}^{-\mathsf{T}}\boldsymbol{f} = \begin{pmatrix} \boldsymbol{F} \\ c_S + \lambda \\ \boldsymbol{M}_S \end{pmatrix}$$

Linear momentum

Scaling

Angular momentum

$$\begin{cases} m\dot{\boldsymbol{v}} = \boldsymbol{F} \\ \frac{1}{2}\,\mathrm{Tr}\,\boldsymbol{\Theta}\dot{\nu} - \frac{1}{|A|^2}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{\Theta}\boldsymbol{\omega} = c_S + \lambda \\ \boldsymbol{\Theta}\dot{\boldsymbol{\omega}} + \frac{1}{|A|^2}(\nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}})\boldsymbol{\Theta}\boldsymbol{\omega} = \boldsymbol{M}_S \end{cases}, \quad \begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v} \\ \dot{A} = \frac{1}{2|A|^2}A(\nu, \boldsymbol{\omega}) \end{cases}$$

Equations of motion in generalized coordinates

$\lambda = 0$: System with 7 degrees of freedom

Rigid Body

- DAE formulation of the rigid body dynamics

$$\begin{cases} m\dot{\boldsymbol{v}} = \boldsymbol{F} \\ \frac{1}{2}\operatorname{Tr}\boldsymbol{\Theta}\dot{\nu} - \frac{1}{|A|^2}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{\Theta}\boldsymbol{\omega} = c_S + \lambda \\ \boldsymbol{\Theta}\dot{\boldsymbol{\omega}} + \frac{1}{|A|^2}(\nu\boldsymbol{I} + \tilde{\boldsymbol{\omega}})\boldsymbol{\Theta}\boldsymbol{\omega} = \boldsymbol{M}_S \\ |A|^2 - 1 = 0 \end{cases}, \quad \begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v} \\ \dot{A} = \frac{1}{2|A|^2}A(\nu, \boldsymbol{\omega}) \end{cases}$$

7 Coordinates
7 Velocities
1 Constraint force

- ODE formulation of the rigid body dynamics

$$\Downarrow \quad \not\Uparrow$$

$$s = |A|^2 = 1 \quad \Rightarrow \quad \dot{s} = \nu = 0 \quad \Rightarrow \quad \ddot{s} = \dot{\nu} = 0$$

$$\begin{cases} m\dot{\boldsymbol{v}} = \boldsymbol{F} \\ \boldsymbol{\Theta}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\boldsymbol{\Theta}\boldsymbol{\omega} = \boldsymbol{M}_S \end{cases}, \quad \begin{cases} \dot{\boldsymbol{r}} = \boldsymbol{v} \\ \dot{A} = \frac{1}{2}A(0, \boldsymbol{\omega}) \end{cases}$$

7 Coordinates
6 Velocities

$$\lambda = -\frac{1}{|A|^2}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{\Theta}\boldsymbol{\omega} - c_S$$

Constraint force preventing the body from scaling

Normal Cone Inclusion

- Equation of motion of a rigid body (DAE)

$$\boldsymbol{M}\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{u}, \boldsymbol{q}, t) - \boldsymbol{W}\lambda = 0$$

$$\boldsymbol{Q}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{u}$$

$$g(\boldsymbol{q}) = |A|^2 - 1 = 0$$

- Normal cone inclusion

$$g(\boldsymbol{q}) = 0 \quad \Longleftrightarrow \quad g(\boldsymbol{q}) \in \mathcal{N}_{\mathbb{R}}(-\lambda)$$

- Equation of motion of a rigid body as differential inclusion

$$\boldsymbol{M}\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{u}, \boldsymbol{q}, t) - \boldsymbol{W}\lambda = 0$$

$$\boldsymbol{Q}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{u}$$

$$g(\boldsymbol{q}) \in \mathcal{N}_{\mathbb{R}}(-\lambda)$$

Conclusions

- Equations of motion of a body with the full degrees of freedom created by quaternion kinematics
- Interpretation of the quaternion unit length restriction as a perfect mechanical constraint in the form of a normal cone inclusion
- Equations of motion of a rigid body formulated as DAE
  - Nonsingular and mechanically correct mass matrix

$$\boldsymbol{M} := \begin{pmatrix} m\boldsymbol{I} & 0 & 0 \\ 0 & \frac{1}{2}\operatorname{Tr}\boldsymbol{\Theta} & 0 \\ 0 & 0 & \boldsymbol{\Theta} \end{pmatrix}$$

  - Singularity-free coordinates
  - Unit length restriction of the quaternion as equation in the DAE
  - Useful for energy consistent integrators

Thank you

Kinematics: Generalized Velocities

- Velocity as quaternion

$$(0, \boldsymbol{\xi}) = A(0, \boldsymbol{\varrho})\tilde{A} + (0, \boldsymbol{r})$$

$$\begin{aligned}
(0, \dot{\boldsymbol{\xi}}) &= \dot{A}(0, \boldsymbol{\varrho})\tilde{A} + A(0, \boldsymbol{\varrho})\dot{\tilde{A}} + (0, \dot{\boldsymbol{r}}) \\
&= \frac{A\tilde{A}}{|A|^2}\dot{A}(0, \boldsymbol{\varrho})\tilde{A} + A(0, \boldsymbol{\varrho})\dot{\tilde{A}}\frac{A\tilde{A}}{|A|^2} + (0, \dot{\boldsymbol{r}}) \\
&= \frac{A}{|A|}\left((0, \boldsymbol{\varrho})\dot{\tilde{A}}A - \tilde{A}\dot{A}(0, -\boldsymbol{\varrho})\right)\frac{\tilde{A}}{|A|} + (0, \dot{\boldsymbol{r}}) \\
&= \frac{A}{|A|}\operatorname{Im}\left((0, \boldsymbol{\varrho})(2\tilde{A}\dot{A})^{\sim}\right)\frac{\tilde{A}}{|A|} + (0, \dot{\boldsymbol{r}}) \\
&= \frac{A}{|A|}(0, \boldsymbol{\varrho}\nu - \tilde{\boldsymbol{\varrho}}\boldsymbol{\omega})\frac{\tilde{A}}{|A|} + (0, \boldsymbol{v}) \\
&= (0, \boldsymbol{v} + \boldsymbol{R}\boldsymbol{\varrho}\nu - \boldsymbol{R}\tilde{\boldsymbol{\varrho}}\boldsymbol{\omega})
\end{aligned}$$

$$\dot{\boldsymbol{\xi}} = (\boldsymbol{I} \quad \boldsymbol{R}\boldsymbol{\varrho} \quad -\boldsymbol{R}\tilde{\boldsymbol{\varrho}})\boldsymbol{Q}\dot{\boldsymbol{q}}$$

- Velocity as vector $\dot{\boldsymbol{\xi}} = \boldsymbol{v} + \boldsymbol{R}\boldsymbol{\varrho}\nu - \boldsymbol{R}\tilde{\boldsymbol{\varrho}}\boldsymbol{\omega}$

$$\begin{pmatrix} \boldsymbol{v} \\ \nu \\ \boldsymbol{\omega} \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & 2a_0 & 2\boldsymbol{a}^\mathsf{T} \\ 0 & -2\boldsymbol{a} & 2a_0\boldsymbol{I} - 2\tilde{\boldsymbol{a}} \end{pmatrix}}_{=:\boldsymbol{Q}} \dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{\xi}} = (\boldsymbol{I} \quad \boldsymbol{R}\boldsymbol{\varrho} \quad -\boldsymbol{R}\tilde{\boldsymbol{\varrho}})\boldsymbol{Q}\dot{\boldsymbol{q}}$$

absolute velocity of a point of the body

Quaternions and Matrices

- Representation as 4x4 matrices

$$\varphi : \mathbb{H} \to \mathbb{R}^{4x4}, \quad \varphi((a_0, \boldsymbol{a})) = \begin{pmatrix} a_0 & -\boldsymbol{a}^\mathsf{T} \\ \boldsymbol{a} & a_0 \boldsymbol{I} + \tilde{\boldsymbol{a}} \end{pmatrix}$$

  - Conjugation corresponds to transposition $\varphi(\tilde{A}) = \varphi(A)^\mathsf{T}$
  - Multiplication with matrices $\varphi(AB) = \varphi(A)\varphi(B)$

- Representation as 4D vectors

$$\psi : \mathbb{H} \to \mathbb{R}^4, \quad \psi((a_o, \boldsymbol{a})) = \begin{pmatrix} a_0 \\ \boldsymbol{a} \end{pmatrix}$$

  - Conjugation $\psi(\tilde{A}) = \boldsymbol{T}\,\psi(A), \quad \boldsymbol{T} = \begin{pmatrix} 1 & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix}$
  - Multiplication with 4x4 matrix $\psi(AB) = \varphi(A)\psi(B)$