Gradient damage models and their use to approximate brittle fracture

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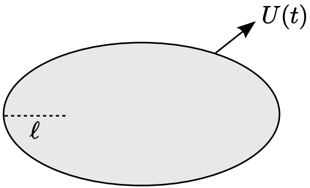
Griffith criterion as unilateral minimization problem

Principal ingredients:

- $\mathcal{K}$ the **pre-assigned path** and ℓ the coordinate of the tip of the crack along this path.
- $\mathcal{P}(\ell)$ the potential energy as a function of ℓ.
- $\mathcal{G}(\ell) = -\partial\mathcal{P}(\ell)/\partial\ell$ the associated energy release rate.
- Surface energy $G_c\,\ell$, proportional to the crack length.



Griffith propagation criterion

$$\ell \geq \ell_0, \qquad \mathcal{G}(\ell) := -\partial\mathcal{P}(\ell)/\partial\ell \leq G_c, \qquad (\ell-\ell_0)(\mathcal{G}(\ell)-G_c) = 0$$

Variational formulation

$$\min_{\ell \geq \ell_0} \mathcal{P}(\ell) + G_c\,\ell$$

Limits: Initiation, Crack paths, Brutal crack propagation

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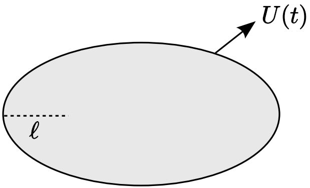

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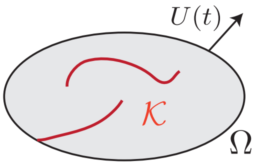
Variational approach of fracture: Griffith functional

Hypotheses:

- Crack ($\mathcal{K}$): surface of discontinuity of the displacements u
- Linear isotropic material, geometrically linear theory.
- Loading: imposed displacement $U(t)$.



Energy functional

$$\mathcal{E}(u,\mathcal{K}) = \underbrace{\int_{\Omega\setminus\mathcal{K}} \frac{1}{2} A_0\, \varepsilon(u)\cdot\varepsilon(u)}_{\textit{Potential energy}} + \underbrace{G_c\, \text{area}(\mathcal{K})}_{\textit{Surface energy}}$$

(A_0 elasticity tensor, ε linearized deformations)

- Principle of least energy for quasi-static evolution (Global energy minimization)

References:

- Francfort and Marigo *J. Mech. Phys. Solids* 1998
- Bourdin, Francfort and Marigo *J. Mech. Phys. Solids* 2000, *J. Elasticity* 2007
- Del Piero, Lancioni and Mach *J. Mech. Phys. Solids* 2007

Variational approach: time-discrete quasi-static evolution problem

Variational formulation for monitonically increasing loadings $(U_1, ...U_i, ...U_n)$

- The state at time $T_{i+1} = T_i + \Delta T$ is:

$$\min_{u \in \mathcal{U}_i(\mathcal{K}),\, \mathcal{K} \supseteq \mathcal{K}_i} \mathcal{E}(u, \mathcal{K})$$

where $(u_i, \mathcal{K}_i)$ is the state at time T_i

$$\mathcal{E}(u, \mathcal{K}) = \int_{\Omega \setminus \mathcal{K}} \frac{1}{2} A_0 \varepsilon(u) \cdot \varepsilon(u) + G_c \, \text{area}(\mathcal{K})$$

- The irreversibility condition $\mathcal{K} \supseteq \mathcal{K}_i$ is fundamental.

This is a Free Discontinuity Problem

- Difficulty : to manage displacement fields u wich can be discontinuous anywhere
- Existence results available in suitable functional setting (SBV/SBD/GSBV/GSBD spaces)

Regularized formulation: Non-local damage models

The energy functional (Griffith)

$$\mathcal{E}(u,\mathcal{K}) = \int_{\Omega\setminus\mathcal{K}} \frac{1}{2} A_0 \varepsilon(u)\cdot\varepsilon(u) + G_c\,\text{area}(\mathcal{K})$$

is approximated, in the sense of Γ-convergence, by a family of regularized elliptic functionals.

A possible regularized functional is

$$\mathcal{E}_\ell(u,\alpha) = \underbrace{\frac{1}{2}\int_\Omega A(\alpha)\,\varepsilon(u)\cdot\varepsilon(u)}_{\text{Appr. Elastic energy}} + G_c \underbrace{\int_\Omega \left(\ell\,\nabla\alpha\cdot\nabla\alpha + \frac{w(\alpha)}{\ell}\right)}_{\text{Appr. crack area}}$$

where α is an additional scalar field and ℓ a numerical parameter.

With suitable choices of the function $w(\alpha)$ and $A(\alpha)$ for $\ell\to 0$

$$\min \mathcal{E}_\ell(u,\alpha) \to \min \mathcal{E}(u,\mathcal{K})$$

(Γ**-convergence** results: convergence of **global** minima)

Regularized formulation: mechanical interpretation

Regularized functional

$$\mathcal{E}_\ell(u,\alpha) = \underbrace{\frac{1}{2}\int_\Omega A(\alpha)\,\varepsilon(u)\cdot\varepsilon(u)}_{\text{Elastic energy}} + \underbrace{G_c\int_\Omega\left(\ell\,\nabla\alpha\cdot\nabla\alpha + \frac{w(\alpha)}{\ell}\right)}_{\text{Dissipated energy}}$$

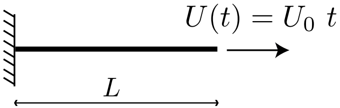
Mechanical interpretation:

- α, a scalar field on Ω: an internal variable representing the damage field.
- $w(\alpha)$: internal dissipation for homogeneous damage processes, e.g. $w(\alpha) = w_1\alpha^2$.
- $A(\alpha)$: the damaged elastic tensor, e.g. $A(\alpha) = A_0\,(1-\alpha)^2$.
- ℓ: the internal length.

The regularized formulation corresponds to
the approximation of the **brittle fracture** problem by
a suitable **non-local damage model** with internal length ℓ.

Regularized damage model: 1D traction problem

Imposed end-displacement $U(t)$, linearized elasticity, brittle isotropic material, quasi-static



$$\underbrace{\mathcal{E}_\ell(u_t, \alpha_t)}_{\text{Total energy}} = \underbrace{\int_0^L \frac{1}{2} E(\alpha_t(x))\, u_t'(x)^2 dx}_{\text{Elastic energy}} + \underbrace{\int_0^L \left(\frac{w(\alpha_t(x))}{\ell} + w_1 \ell\, \alpha_t'(x)^2 \right) dx}_{\text{Dissipated energy}}$$

Unilateral local minimality

We define the damage evolution (u_t, α_t) through an unilateral (local) minimality principle on the total energy under the irreversibility condition:

$$\dot{\alpha}_t(x) := \frac{d\alpha_t(x)}{dt} \in \mathcal{D} = \{\beta \in H^1(\Omega) : \beta(x) \geq 0 \text{ for almost all } x\}$$

Rate evolution problem

$$\mathcal{C}(U_t) = \left\{v \in H^1(\Omega) : v(0) = 0,\, v(L) = U_t\right\},\ \mathcal{D} = \left\{\beta \in H^1(\Omega) : \beta(x) \geq 0 \text{ for almost all } x\right\}$$

For each $t > 0$, find (u_t, α_t) in $\mathcal{C}(U_t) \times \mathcal{D}_1$ such that

$$(\dot{u}_t, \dot{\alpha}_t) \in \mathcal{C}(\dot{U}_t) \times \mathcal{D} \quad \text{and} \quad \forall (v, \beta) \in \mathcal{C}(\dot{U}_t) \times \mathcal{D},$$

$$D\mathcal{E}_\ell(u_t, \alpha_t)(v - \dot{u}_t, \beta - \dot{\alpha}_t) \geq 0$$

Equilibrium equations (obtained by setting $\beta = \dot{\alpha}_t$)

$$\sigma_t'(x) = 0, \qquad \sigma_t(x) = E(\alpha_t(x))u_t'(x), \quad \forall x \in (0, L).$$

Damage problem (obtained by setting $v = \dot{u}_t$)

- Irreversibility : $\dot{\alpha}_t \geq 0, \quad \alpha_0 = 0,$
- Damage criterion : $-w_1 \ell^2 \alpha_t'' + \frac{1}{2} E'(\alpha_t) {u_t'}^2 + w'(\alpha_t) \geq 0,$
- Energy balance : $\dot{\alpha}_t \left(-w_1 \ell^2 \alpha_t'' + \frac{1}{2} E'(\alpha_t) {u_t'}^2 + w'(\alpha_t) \right) = 0,$
- Boundary conditions : $\alpha_t'(0) \leq 0, \quad \alpha_t'(L) \geq 0.$

Second order optimality conditions: stability

Stability criterion

The state (u_t, α_t) is stable at time t
iff
(u_t, α_t) is a unilateral local minimum of the total energy

$\exists h > 0$ such that $\mathcal{E}_\ell(u_t + hv, \alpha_t + h\beta) \geq \mathcal{E}_\ell(u_t, \alpha_t), \qquad \forall (v, \beta) \in \mathcal{C}_1 \times D$

Only reachable damage states are tested *i.e.* $\alpha_t + h\beta$ with $\beta \geq 0$

By Taylor expansion, the stability is assessed by studying the sign of the second derivative

$$\mathcal{E}_\ell(u_t + hv, \alpha_t + h\beta) - \mathcal{E}_\ell(u_t, \alpha_t) = h\mathcal{E}'_\ell(u_t, \alpha_t)(v, \beta) + \frac{1}{2}h^2\mathcal{E}''_\ell(u_t, \alpha_t)(v, \beta) + \ldots$$

Introducing the Rayleigh ratio

$$\mathcal{R}_t(v, \beta) = \frac{\int_0^L w_1 \ell^2 \beta'^2 \, dx + \int_0^L E(\alpha_t) \left(v' + \frac{E'(\alpha_t)}{E(\alpha_t)} t\beta \right)^2 dx}{\int_0^L \left(\frac{1}{2} S''(\alpha_t)\sigma_t^2 - w''(\alpha_t) \right) \beta^2 \, dx}$$

a sufficient (*resp.* necessary) condition for stability is that

$$\varrho = \min_{(v,\beta) \in \mathcal{C}_0 \times \mathcal{D}} \mathcal{R}_t(v, \beta) > (\text{resp. } \geq) 1.$$

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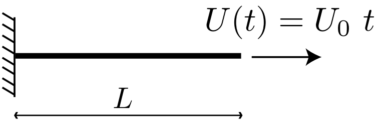
Homogeneous solutions



Homogeneous solutions of the evolution problem:

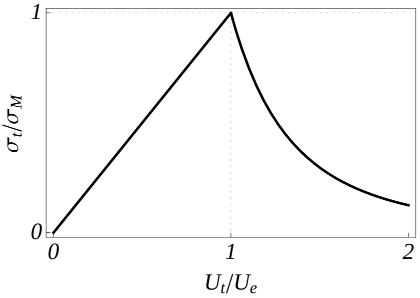
$$\begin{cases} u_t(x,t) = \varepsilon(t)\ x, \quad \varepsilon(t) = \dfrac{U_0}{L} t \\ \alpha(x,t) = \alpha(t) \end{cases}$$

- Force-displacement (σ-ε) diagram?
- Is there an elastic phase ($\alpha(t) = 0$) for $t \leq t_e$? Which is the elastic limit stress σ_e?
- Is the homogeneous solution stable? When? Is there a maximum allowable stress σ_M for homogeneous solutions?

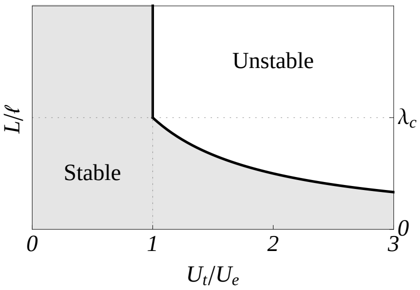
$$\mathcal{E}_\ell(u_t,\alpha_t)=\int_0^L \frac{1}{2}E(\alpha_t(x))\,u_t'(x)^2dx+\int_0^L\left(\frac{w(\alpha_t(x))}{\ell}+w_1\ell\,\alpha_t'(x)^2\right)dx$$

$$E(\alpha)=E_0(1-\alpha)^2,\quad w(\alpha)=w_1\alpha$$

Stress-displacement diagram



Stability diagram



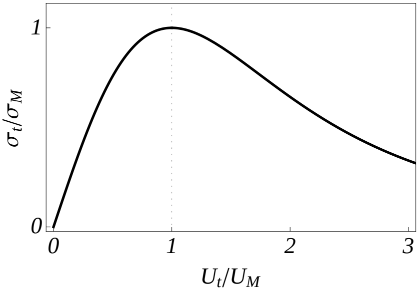
$$\sigma_e=\sigma_M=\sqrt{w_1E_0},\qquad U_e=\sqrt{\frac{w_1}{E_0}}L=\frac{\sigma_M}{E_0}L. \tag{1}$$

Homogeneous solutions: example 2

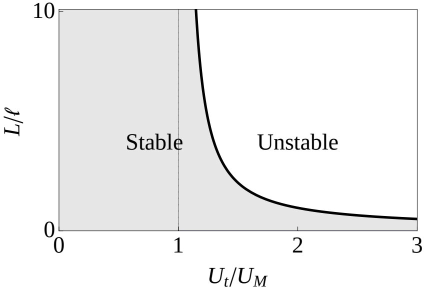
$$\mathcal{E}_\ell(u_t,\alpha_t) = \int_0^L \frac{1}{2} E(\alpha_t(x))\, u_t'(x)^2 dx + \int_0^L \left(\frac{w(\alpha_t(x))}{\ell} + w_1 \ell\, \alpha_t'(x)^2 \right) dx$$

$$E(\alpha) = E_0(1-\alpha)^2, \quad w(\alpha) = w_1 \alpha^2$$

Stress-displacement diagram



Stability diagram



$$U_M = \frac{16\sigma_M}{9E_0} L, \qquad \sigma_M = \frac{3\sqrt{3}}{8\sqrt{2}} \sqrt{w_1 E_0}, \qquad \alpha_t = \frac{U_t^2}{U_t^2 + 3U_M^2}. \tag{2}$$

Localized solutions: fracture as localized damage

Solution with a single fully developed localization inside the bar for the case

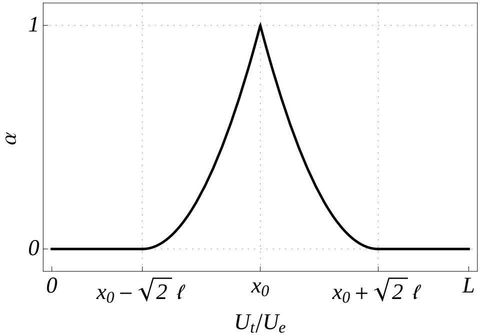
$$E(\alpha) = E_0(1-\alpha)^2, \quad w(\alpha) = w_1\alpha$$

Damage profile

$$\alpha(x) = \left(1 - \frac{|x-x_0|}{\sqrt{2}\ell}\right)^2$$

for

$$x \in [x_0 - \sqrt{2}\ell, x_0 + \sqrt{2}\ell],$$



The energy dissipated in this kind of solution is

$$G_c = \frac{4\sqrt{2}}{3} w_1 \ell$$

This gives a relation between the volume dissipation w_1 and the surface dissipation G_c.

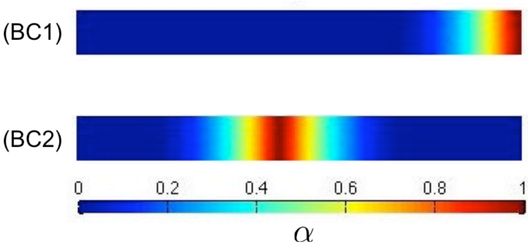
Solution algorithm based on an **alternate minimization**

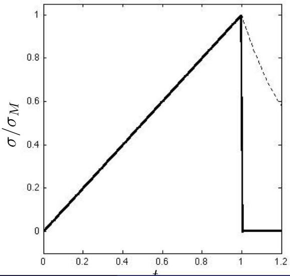
1. **Initialization**
   1. Set the values of the key numerical parameters: ℓ, the mesh size, ΔT, the residual stiffness, δ.
   2. Set $k = 0$ and $(u^0_{i+1}, \alpha^0_{i+1}) := (u_i, \alpha_i)$.
2. **Interation** k
   1. **Equilibrium** problem:
   $$u^k_{i+1} := \arg\min_u \mathcal{E}_\ell(u, \alpha^{k-1}_{i+1})$$
   under the constraint $u = u_0$ on $\partial_u\Omega$.
   2. **Damage** problem:
   $$\alpha^k_{i+1} := \arg\min_\alpha \mathcal{E}_\ell(u^k_{i+1}, \alpha)$$
   under the constraint $\alpha_i \leq \alpha \leq 1$.
3. **End**
   1. Repeat step 2 until $\|\alpha_{i+1} - \alpha_i\|_\infty \leq \delta$.
   2. Set $(u_{i+1}, \alpha_{i+1}) := (u^k_{i+1}, \alpha^k_{i+1})$.

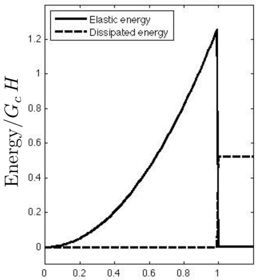
- Finite elements (unstructured, uniform meshes): the mesh size h much be smaller than ℓ
- The step (2.2) implies a bound-constrained minimization of a quadratic functional of α.
- Damage is treated as a **nodal** variable.

2D numerical simulation: long bar

$$E(\alpha) = E_0(1-\alpha)^2, \quad w(\alpha) = w_1\alpha$$

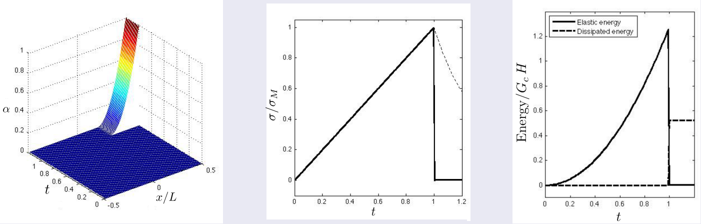




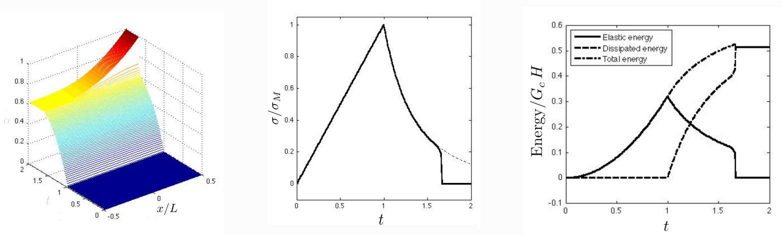


Long vs short bars: scale effect

Long bars ($L = 2\,\lambda_c \ell$)



Short bars ($L = \lambda_c \ell\,/2$)



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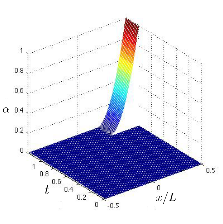

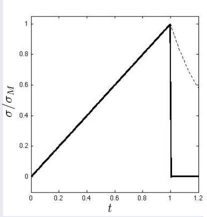

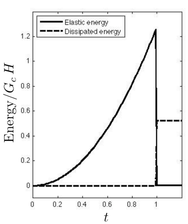

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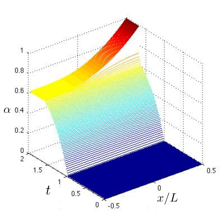
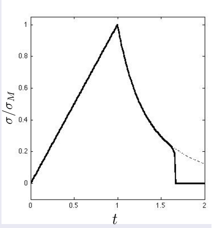
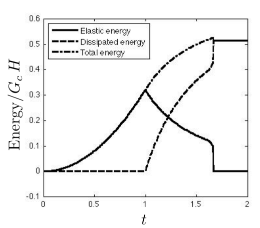


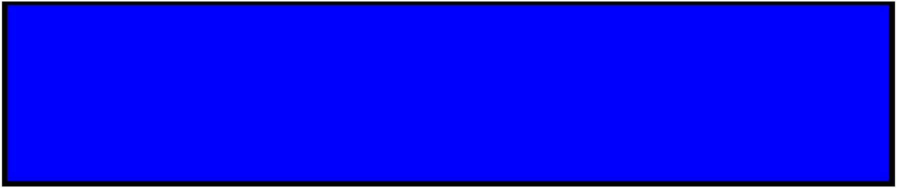




Numerical results: thermal shock cracks

$$\text{Width} = 5, \quad \text{Thickness} = 1, \quad \text{Element size} = .01$$

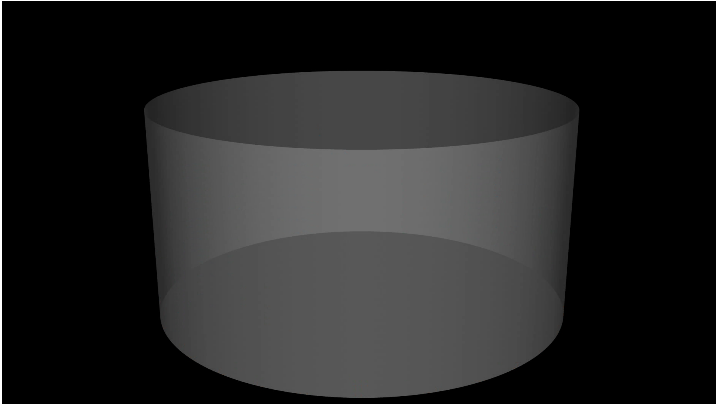
$$\ell = .02, \quad \theta_0 = 54$$



- No initial cracks
- No assumptions on periodicity
- No assumptions on crack pattern

Cylinder
(Thermal shock on the bottom face, free on the boundary)

891 000 elements, 101 time steps. Approx. **6h** walltime on **256 cpus** (Ranger, TACC)



Only fractures ($\alpha > 0.9$) are reported. Colors are for the temperature field.

B.Bourdin, C.Maurini and M.Knepley (in preparation)