On the stability of quasi-static paths for elastic-plastic systems with hardening

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Abstract

This is a brief text based on the communication to the seventh Meeting "Unilateral Problems in Structural Analysis", held in Palmanova, Italia, June 17-19, 2010.

1 Introduction

The focus of this talk is on the concept of stability of quasi-static paths as introduced by J. A. C. Martins and co-authors, in works ranging from 2000 to 2008.
Take a dynamic problem

$$m\ddot{q} = f(t, q, \dot{q}) + R$$

and the corresponding quasi-static problem

$$0 = f(t, q, \dot{q}) + R.$$

Replacing dynamics by quasi-statics, for slow evolutions, requires:
- mathematical justification and the
- assessment of the limits of validity.

In a CISM course on instabilities (Udine, 1999; published 2000), Martins, Loret and Simões recognized the role of different time scales, in the consideration of dynamical and quasi-static problems, as well as the nature of a singular perturbation problem, and this approach was developed subsequently. (Investigation of the approximation between such problems includes earlier works by J. Martins, F. Gastaldi and M. M. Marques).

The general idea of the present study is the following:
besides the variable t, the (fast) physical time, one considers also

$$\lambda = \varepsilon t,$$

a (slow) loading parameter.
If one lets the rate of loading tend to zero

$$\varepsilon \to 0,$$

a quasi-static trajectory is expected to come up in the limit.

Stability of a quasistatic trajectory is taken to mean that dynamic trajectories starting close to it will remain close, if the loading is slow enough.

The plan of the talk is to consider essentially the case of

Sect. 2 - A system with a finite number of degrees-of-freedom,

as in the paper
J.A.C. Martins, M.D.P. Monteiro Marques and A. Petrov, On the stability of quasi-static paths for finite-dimensional elastic-plastic systems with hardening, ZAMM 87 (2007) 303-313,

and then to mention very briefly

Sect. 3 - The case of an elastic-plastic 1-D bar,

as in
J.A.C. Martins, M.D.P. Monteiro Marques and A. Petrov, On the stability of elastic-plastic systems with hardening, J. Math. Anal. Appl. 343 (2008) 1007–1021,
and

Sect. 4 - A generalization, applicable to 3-D,

as in
A. Mielke, A. Petrov and J.A.C. Martins, Convergence of solutions of kinetic variational inequalities in the rate-independent quasi-static limit, J. Math. Anal. Appl. 348 (2008) 1012–1020.

Before proceeding to Sect. 2, a few comments:

1 - Recall the equation of dynamics:

$$m\ddot{q}(t) = f(t, q(t), \dot{q}(t)) + R(t, q(t), \dot{q}(t))$$

and the loading parameter $\lambda = \varepsilon t$. Taking

$$\bar{q}(\lambda) = q(\frac{\lambda}{\varepsilon}) = q(t),$$

consideration of the above equation at $t = \lambda/\varepsilon$ yields

$$m\varepsilon^2 \bar{q}''(\lambda) = f(\lambda/\varepsilon, \bar{q}(\lambda), \epsilon\bar{q}'(\lambda)) + R(\lambda/\varepsilon, \,.\,, \,.\,),$$

where $'$ denotes differentiation with respect to λ.
Thus, the coefficient of the higher-order term (i.e. the "mass") $m\varepsilon^2 \to 0$, highlighting the singular perturbation and quasi-static nature of the limiting procedure.
Notice also that the presence of λ/ε in the r.h.s. may require a special structure of the given forces or of the reaction law for R.

2 - Our topic is stability, with a few differences:

- The comparison is made between dynamic solutions and a quasi-static trajectory, which is not a true dynamic solution.

- Consideration of different time-scales and/or of different initial values is a mechanically relevant novelty with respect to previous studies (e.g. Duvaut-Lions).

- Even in the absence of damping, the notion of stability is extended to allow for dynamic trajectories remaining close to q.s. paths.

2 A system with a finite number of degrees-of-freedom

We consider a finite dimensional model-problem involving an elastic-plastic structure with linear kinematic hardening, with a geometrically linear behaviour. The model is a plane truss composed of n elastic-plastic bars, meeting at a number of nodes (hinges), such that N is the total number of free nodal x- and y-displacement components, i.e. all those that are not fixed. We group these nodal displacement components u_I, $I = 1, \ldots, N$, in the nodal displacement vector $\boldsymbol{u}$. As a consequence of the geometric linearity, the elongations e_i, $i = 1, \ldots, n$, of the elastic-plastic bars are linearly related to the nodal displacements, i.e.

$$\boldsymbol{e} = \boldsymbol{L}\boldsymbol{u}, \tag{1}$$

where $\boldsymbol{e}$ is the n-vector that collects all bar elongations, and $\boldsymbol{L}$ is a $n \times N$ constant kinematic matrix, assumed to be injective.
Each bar (labeled by i) is assumed to have an elastic-plastic behaviour with kinematic hardening, described by the following elements:

E_i and H_i denote the stiffness of the elastic spring (in series with the plastic/sliding element) and of the hardening spring (the one in parallel with the plastic element), respectively. The elongation e_i of each bar can be decomposed into elastic and plastic contributions, $e_i = e_i^e + e_i^p$, or, defining the n-vectors $\boldsymbol{e}^e$ and $\boldsymbol{e}^p$:

$$\boldsymbol{e} = \boldsymbol{e}^e + \boldsymbol{e}^p. \tag{2}$$

Denoting by σ_i the force in the bar i, that force is, on one hand, proportional to the elastic elongation e_i^e (Hooke's law) $\sigma_i = E_i e_i^e = E_i(e_i - e_i^p)$ and, on the other hand, it equals the sum of the force in the plastic element, r_i, with the force in the hardening spring, $\sigma_i = r_i + H_i e_i^p$. We collect the stiffness coefficients E_i and H_i in the diagonal $n \times n$ matrices $\boldsymbol{E}$ and $\boldsymbol{H}$, and the forces σ_i and r_i in the n-vectors $\boldsymbol{\sigma}$ and $\boldsymbol{r}$.

It can be shown that the forces in the elastic-plastic bars take the form:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{r}) = \boldsymbol{D}(\boldsymbol{L}\boldsymbol{u} + \boldsymbol{H}^{-1}\boldsymbol{r}) = \boldsymbol{D}\boldsymbol{L}\boldsymbol{u} + \boldsymbol{E}\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}, \tag{3}$$

where $\boldsymbol{D} = \left(\boldsymbol{E}^{-1} + \boldsymbol{H}^{-1}\right)^{-1}$ and $\widetilde{\boldsymbol{D}} = \boldsymbol{E} + \boldsymbol{H}$ are diagonal matrices.

As for the forces $\boldsymbol{f}$ we decompose them into external forces applied on the nodes, $\boldsymbol{f}_{\text{ext}}$, and internal forces, $\boldsymbol{f}_{\text{int}}$, acted on the nodes by the neighboring bars, with

$$\boldsymbol{f} = \boldsymbol{f}_{\text{ext}}(\lambda) + \boldsymbol{f}_{\text{int}}(\boldsymbol{u}, \boldsymbol{r}).$$

Indeed, it can be shown that

$$\boldsymbol{f}_{\text{int}} = \boldsymbol{f}_{\text{int}}(\boldsymbol{u}, \boldsymbol{r}) = -\boldsymbol{L}^T \boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{r}). \tag{4}$$

Each plastic element is characterized by the inequalities and flow rule:

$$|r_i| \leq 1, \ \frac{de_i^p}{d\lambda} \begin{cases} \geq 0 \text{ if } r_i = +1, \\ = 0 \text{ if } \ -1 < r_i < +1, \\ \leq 0 \text{ if } r_i = -1, \end{cases} \quad \forall i = 1, \ldots, n, \tag{5}$$

where, w.l.o.g., the yield forces in all bars, both in tension and compression, are assumed to be unitary. This may also be written in inclusion form:

$$\frac{de_i^p}{d\lambda} \in N_{[-1,1]}(r_i).$$

DYNAMICS

We write the governing dynamic equations of the system in terms of a load parameter λ that parametrizes the evolution of the given external forces $\boldsymbol{f}_{\text{ext}} = \boldsymbol{f}_{\text{ext}}(\lambda)$.
It is related to the physical time t according to

$$\lambda = \lambda_1 + \varepsilon t,$$

λ_1 being the initial value of the load parameter. The time rate of change of the load $\varepsilon = d\lambda/dt$ is supposed to be small and $(\cdot)'$ denotes differentiation with respect to λ, $d(\cdot)/d\lambda$. Taking $\boldsymbol{M}$ as the (constant, symmetric, positive definite) mass matrix, the governing dynamic equations can be written as

$$\varepsilon^2 \boldsymbol{M}\boldsymbol{u}''(\lambda) = \boldsymbol{f}_{\text{ext}}(\lambda) + \boldsymbol{f}_{\text{int}}(\boldsymbol{u}(\lambda), \boldsymbol{r}(\lambda)),$$

or more precisely

$$\varepsilon^2 \boldsymbol{M}\boldsymbol{u}''(\lambda) + \boldsymbol{L}^T \boldsymbol{D}\boldsymbol{L}\boldsymbol{u}(\lambda) + \boldsymbol{L}^T \boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}(\lambda) = \boldsymbol{f}_{\text{ext}}(\lambda). \tag{6}$$

To this equation must be added the flow rule:

$$(\boldsymbol{e}^p)' \in N_{\mathcal{C}}(\boldsymbol{r}),$$

where $\mathcal{C} = [-1, 1]^n$, $\boldsymbol{e}^p = (e_i^p)$ and $\boldsymbol{r} = (r_i)$.

Taking into account the relations between quantities above and introducing the velocity

$$\boldsymbol{v} = \varepsilon \boldsymbol{u}',$$

we obtain the system

$$\begin{cases} \varepsilon \boldsymbol{u}' - \boldsymbol{v} = 0, \\ \varepsilon \boldsymbol{M}\boldsymbol{v}' + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{u} + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r} = \boldsymbol{f}_{\text{ext}}, \\ \widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}' - \boldsymbol{H}^{-1}\boldsymbol{D}\boldsymbol{L}\boldsymbol{u}' + \boldsymbol{N}_{\mathcal{C}}(\boldsymbol{r}) \ni 0, \end{cases} \tag{7}$$

with some initial conditions

$$\Big(\boldsymbol{u}(\lambda_1), \boldsymbol{v}(\lambda_1), \boldsymbol{r}(\lambda_1)\Big) = (\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{r}_1) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}. \tag{8}$$

The corresponding quasi-static system is (take $\varepsilon = 0$ above):

$$\begin{cases} \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\bar{\boldsymbol{u}} + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\bar{\boldsymbol{r}} = \boldsymbol{f}_{\text{ext}}, \\ \widetilde{\boldsymbol{D}}^{-1}\bar{\boldsymbol{r}}' - \boldsymbol{H}^{-1}\boldsymbol{D}\boldsymbol{L}\bar{\boldsymbol{u}}' + \boldsymbol{N}_{\mathcal{C}}\Big(\bar{\boldsymbol{r}}\Big) \ni 0, \end{cases} \tag{9}$$

with initial conditions

$$\bar{\boldsymbol{r}}(\lambda_1) = \bar{\boldsymbol{r}}_1 \in \mathcal{C}. \tag{10}$$

A few remarks:
Note that, consistently with the above, the quasi-static displacement rate with respect to the physical time vanishes ($\bar{\boldsymbol{v}} \equiv 0$). Also, the first (vector) equation gives $\bar{\boldsymbol{u}}$ in terms of $\bar{\boldsymbol{r}}$, so that the initial value may be specified only for the latter. Given the presence of a differential inclusion associated to a maximal monotone operator, this problem exhibits a certain level of non-smoothness.

The above differential inclusions may also be written in VARIATIONAL FORM:

(Dynamics) $\boldsymbol{r} \in \mathcal{C}$ and

$$\int_{\lambda_1}^{\lambda} \Big(\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}' - \boldsymbol{H}^{-1}\boldsymbol{D}\boldsymbol{L}\boldsymbol{u}'\Big) \cdot (\boldsymbol{r}^* - \boldsymbol{r})\, d\xi \geq 0, \, \forall \boldsymbol{r}^* \in \mathcal{C}, \, \forall \lambda \in [\lambda_1, \lambda_2], \tag{11}$$

(Quasi-statics) $\bar{\boldsymbol{r}} \in \mathcal{C}$ and

$$\int_{\lambda_1}^{\lambda} \Big(\widetilde{\boldsymbol{D}}^{-1}\bar{\boldsymbol{r}}' - \boldsymbol{H}^{-1}\boldsymbol{D}\boldsymbol{L}\bar{\boldsymbol{u}}'\Big) \cdot (\boldsymbol{r}^* - \bar{\boldsymbol{r}})\, d\xi \geq 0, \, \forall \boldsymbol{r}^* \in \mathcal{C}, \, \forall \lambda \in [\lambda_1, \lambda_2]. \tag{12}$$

By the theory of maximal monotone operators and evolution problems, one is able to prove that:

Theorem 2.1 *If* $\boldsymbol{f}_{\mathrm{ext}}$ *belongs to* $\boldsymbol{W}^{1,\infty}_N(\lambda_1,\lambda_2)$ *and* $\boldsymbol{r}_1 \in \mathcal{C}$*, then there exists a unique solution* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{r})$ *of the dynamics problem (7) together with initial condition* (8)*, belonging to* $\left(\boldsymbol{W}^{1,\infty}_N(\lambda_1,\lambda_2)\right)^2 \times \boldsymbol{W}^{1,\infty}_n(\lambda_1,\lambda_2)$*.*

Theorem 2.2 *If* $\boldsymbol{f}_{\mathrm{ext}}$ *belongs to* $\boldsymbol{W}^{1,\infty}_N(\lambda_1,\lambda_2)$ *and* $\bar{\boldsymbol{r}}_1 \in \mathcal{C}$*, then there exists a unique solution* $\bar{\boldsymbol{r}}$ *of (9) belonging to* $\boldsymbol{W}^{1,\infty}_n(\lambda_1,\lambda_2)$*, with initial condition* $\bar{\boldsymbol{r}}_1$*.*

By (9), it follows that $\bar{\boldsymbol{u}}$ also belongs to $\boldsymbol{W}^{1,\infty}_N(\lambda_1,\lambda_2)$. Later, we shall assume that

$$\boldsymbol{f}_{\mathrm{ext}} \in \boldsymbol{W}^{2,\infty}_N(\lambda_1,\lambda_2)$$

to allow further differentiation of the solutions.

The mathematical definition of stability of a quasi-static path at an equilibrium point $(\bar{u}(\lambda_1), \bar{r}_1)$ is the following, for the dynamic and quasi-static systems under analysis:

Definition. The quasi-static path $(\bar{\boldsymbol{u}}(\lambda), \bar{\boldsymbol{r}}(\lambda))$ is said to be STABLE at λ_1 if there exists $0 < \Delta\lambda \leq \lambda_2 - \lambda_1$, such that, for all $\delta > 0$ there exists $\bar{\rho}(\delta) > 0$ and $\bar{\varepsilon}(\delta) > 0$ such that if $\varepsilon > 0$ and the initial conditions $\boldsymbol{u}_1$, $\boldsymbol{v}_1$, $\boldsymbol{r}_1$, $\bar{\boldsymbol{r}}_1$ (with $\boldsymbol{r}_1 \in \mathcal{C}, \bar{\boldsymbol{r}}_1 \in \mathcal{C}$) are such that

$$|\boldsymbol{v}_1|^2 + |\boldsymbol{u}_1 - \bar{\boldsymbol{u}}(\lambda_1)|^2 + |\boldsymbol{r}_1 - \bar{\boldsymbol{r}}_1|^2 \leq \bar{\rho}(\delta) \text{ and } \varepsilon \leq \bar{\varepsilon}(\delta),$$

then the solution $(\boldsymbol{u}(\lambda), \boldsymbol{v}(\lambda), \boldsymbol{r}(\lambda))$ of the dynamic system (7)–(8) satisfies

$$|\boldsymbol{v}(\lambda)|^2 + |\boldsymbol{u}(\lambda) - \bar{\boldsymbol{u}}(\lambda)|^2 + |\boldsymbol{r}(\lambda) - \bar{\boldsymbol{r}}(\lambda)|^2 \leq \delta, \ \forall \lambda \in [\lambda_1, \lambda_1 + \Delta\lambda].$$

In other words, in order that the two solutions remain close to each other in some finite interval of load, it suffices that the dynamic solution of (7) is initially close to the quasi-static solution of (9) and that the loading rate ε is sufficiently small. The stability follows from

Theorem 2.3 *If the initial data are admissible and* $\boldsymbol{f}_{\text{ext}} \in \boldsymbol{W}_N^{2,\infty}(\lambda_1, \lambda_2)$*, then there exist* $\gamma_i > 0$*,* $i = 1, 2$*, such that*

$$\begin{aligned} &|\boldsymbol{v}(\lambda)|^2 + |\boldsymbol{u}(\lambda) - \bar{\boldsymbol{u}}(\lambda)|^2 + |\boldsymbol{r}(\lambda) - \bar{\boldsymbol{r}}(\lambda)|^2 \\ &\qquad \leq \gamma_1 \left(|\boldsymbol{v}_1|^2 + |\boldsymbol{u}_1 - \bar{\boldsymbol{u}}(\lambda_1)|^2 + |\boldsymbol{r}_1 - \bar{\boldsymbol{r}}_1|^2 \right) + \gamma_2 \varepsilon. \end{aligned} \tag{13}$$

SKETCH OF PROOF.
We subtract the equality in the quasi-static system from the equality in the dynamic problem, then take the scalar product with $\boldsymbol{u}' - \bar{\boldsymbol{u}}'$ and we integrate over (λ_1, λ). On the other hand, we choose $\boldsymbol{r}^* = \bar{\boldsymbol{r}}$ in (11) and $\boldsymbol{r}^* = \boldsymbol{r}$ in (12), and we add (11) to (12). This leads to the following system:

$$\begin{cases} \displaystyle\int_{\lambda_1}^{\lambda} \varepsilon^2 (\boldsymbol{M}\boldsymbol{u}'') \cdot \boldsymbol{u}' \, d\xi + \int_{\lambda_1}^{\lambda} \Big(\boldsymbol{L}^T \boldsymbol{D} \boldsymbol{L} (\boldsymbol{u} - \bar{\boldsymbol{u}}) + \boldsymbol{L}^T \boldsymbol{D} \boldsymbol{H}^{-1} (\boldsymbol{r} - \bar{\boldsymbol{r}}) \Big) \cdot (\boldsymbol{u}' - \bar{\boldsymbol{u}}') \, d\xi \\ \displaystyle = \int_{\lambda_1}^{\lambda} \varepsilon^2 (\boldsymbol{M}\boldsymbol{u}'') \cdot \bar{\boldsymbol{u}}' \, d\xi, \\ \displaystyle\int_{\lambda_1}^{\lambda} \Big(\widetilde{\boldsymbol{D}}^{-1} (\boldsymbol{r}' - \bar{\boldsymbol{r}}') - \boldsymbol{H}^{-1} \boldsymbol{D} \boldsymbol{L} (\boldsymbol{u}' - \bar{\boldsymbol{u}}') \Big) \cdot (\boldsymbol{r} - \bar{\boldsymbol{r}}) \, d\xi \leq 0. \end{cases} \tag{14}$$

Now, in the above expressions, two terms cancel each other, while other terms are the derivatives of the terms in the function

$$h = (\boldsymbol{M}\boldsymbol{v})\cdot\boldsymbol{v} + \Big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}(\boldsymbol{u}-\bar{\boldsymbol{u}})\Big)\cdot(\boldsymbol{u}-\bar{\boldsymbol{u}}) + \Big(\widetilde{\boldsymbol{D}}^{-1}(\boldsymbol{r}-\bar{\boldsymbol{r}})\Big)\cdot(\boldsymbol{r}-\bar{\boldsymbol{r}}) \quad (15)$$

By the Cauchy-Schwarz inequality, we obtain the following inequality

$$h(\lambda) \le h(\lambda_1) + \|\boldsymbol{M}\|_\infty \left(\int_{\lambda_1}^{\lambda} |\varepsilon\boldsymbol{v}'|^2\, d\xi\right)^{1/2} \left(\int_{\lambda_1}^{\lambda} |\bar{\boldsymbol{u}}'|^2\, d\xi\right)^{1/2}, \quad (16)$$

where $\|\boldsymbol{M}\|_\infty = \max_i \sum_j |M_{ij}|$, Observing that $\boldsymbol{M}$, $\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}$ and $\widetilde{\boldsymbol{D}}^{-1}$ are symmetric positive definite matrices, we conclude from (16) that there exists $\alpha > 0$ such that

$$|\boldsymbol{v}(\lambda)|^2 + |\boldsymbol{u}(\lambda) - \bar{\boldsymbol{u}}(\lambda)|^2 + |\boldsymbol{r}(\lambda) - \bar{\boldsymbol{r}}(\lambda)|^2 \le \alpha h(\lambda_1) + \alpha \left(\int_{\lambda_1}^{\lambda} |\varepsilon\boldsymbol{v}'|^2\, d\xi\right)^{1/2} \left(\int_{\lambda_1}^{\lambda} |\bar{\boldsymbol{u}}'|^2\, d\xi\right)^{1/2},$$

so that it is sufficient to control the term $\varepsilon\boldsymbol{v}'$, i.e. $\varepsilon^2\boldsymbol{u}''$.

The a priori bound on $\varepsilon\boldsymbol{v}'$ can be obtained as follows.

First consider the regularized (Yosida-Moreau) problem for $\mu > 0$:

$$\begin{cases} \varepsilon^2 \boldsymbol{M}\boldsymbol{u}_\mu'' + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{u}_\mu + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}_\mu = \boldsymbol{f}_{\text{ext}}, \\ \widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}_\mu' - \boldsymbol{H}^{-1}\boldsymbol{D}\boldsymbol{L}\boldsymbol{u}_\mu' + \dfrac{1}{\mu}\Big(\boldsymbol{r}_\mu - \mathbf{proj}_{\mathcal{C}}\boldsymbol{r}_\mu\Big) = 0, \end{cases} \quad (17)$$

with

$$\Big(\boldsymbol{u}_\mu(\lambda_1), \boldsymbol{v}_\mu(\lambda_1), \boldsymbol{r}_\mu(\lambda_1)\Big) = (\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{r}_1) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}.$$

Here $\mathbf{proj}_{\mathcal{C}}$ denotes the projection on the convex $\mathcal{C}$.

As $\mu \to 0$, the solutions $\boldsymbol{u}_\mu$, $\boldsymbol{r}_\mu$ converge to the solution $\boldsymbol{u}$, $\boldsymbol{r}$ of the dynamical problem.
Extra-regularity, as $\boldsymbol{u}_\mu$ belongs to $W^{3,\infty}$ and $\boldsymbol{r}_\mu$ to $W^{2,\infty}$, allows differentiation of equations and this is helpful to establish energy estimates which will be extendable to $\boldsymbol{u}$ and thus to $\boldsymbol{v} = \varepsilon\boldsymbol{u}'$. Also, Gronwall's lemma may be applied, etc. One finally obtains an estimate for $\varepsilon\boldsymbol{v}'$:

Lemma 2.4 *Assume that $\boldsymbol{r}_1 \in \mathcal{C}$ and $\boldsymbol{f}_{\text{ext}}$ belongs to $\boldsymbol{W}_N^{2,\infty}(\lambda_1, \lambda_2)$. Then there exists a positive constant $c(\lambda_1, \lambda_2)$ which depends on the interval of loading and which is such that*

$$\begin{aligned} |\varepsilon\boldsymbol{v}'(\lambda)|^2 \le c(\lambda_1, \lambda_2)\Big(&|\boldsymbol{v}_1|^2 + |\boldsymbol{u}_1 - \bar{\boldsymbol{u}}(\lambda_1)|^2 + |\boldsymbol{r}_1 - \bar{\boldsymbol{r}}_1|^2 \\ &+ \varepsilon^2|\boldsymbol{f}_{\text{ext}}'(\lambda_1)|^2 + \varepsilon^2\|\boldsymbol{f}_{\text{ext}}'\|^2_{L^\infty(\lambda_1,\lambda_2)} + \varepsilon^2\|\boldsymbol{f}_{\text{ext}}''\|^2_{L^2(\lambda_1,\lambda_2)}\Big). \end{aligned} \quad (18)$$

from which, as explained above, the desired stability follows.

3 The case of an elastic-plastic 1-D bar

The mathematical formulation now requires the use of functions of t or λ and of a (space) variable $x \in [0, L]$. Leaving all details to the paper mentioned in the plan of this talk, the dynamics can be expressed by the system:

$$\begin{cases} \varepsilon u' - v = 0, \\ \varepsilon v' - u_{xx} - r_x = f, \\ u'_x - r' \in N_{\mathcal{C}}(r), \end{cases} \tag{19}$$

together with the Dirichlet boundary conditions $u = v = 0$ on $\{0, L\} \times (\lambda_1, \lambda_2)$, and the initial conditions $\big(v(\lambda_1), u(\lambda_1), r(\lambda_1)\big) = (v_1, u_1, r_1) \in V_0 \times W$. where $\mathcal{C}$ is now the set of functions $r \in L^2([0, L])$ such that $|r| \le 1$ a.e., while $V_0 = H^1_0(0, L)$ and W is a certain subspace of $V_0 \times \mathcal{C}$.

The corresponding quasi-static system is then (let $\varepsilon = 0$ in (19))

$$\begin{cases} -\bar{u}_{xx} - \bar{r}_x = f, \\ \bar{u}'_x - \bar{r}' \in N_{\mathcal{C}}(\bar{r}), \end{cases} \tag{20}$$

with the Dirichlet boundary conditions $\bar{u} = 0$ on $\{0, L\} \times (\lambda_1, \lambda_2)$, and the initial conditions $\big(\bar{u}(\lambda_1), \bar{r}(\lambda_1)\big) = (\bar{u}_1, \bar{r}_1) \in W$.

The definition of stability is analogous to the one in the previous section and the precise mathematical result reads:

Theorem 3.1 *(Stability). Let* $(v_1, u_1, r_1) \in V_0 \times W$*,* $(\bar{u}_1, \bar{r}_1) \in W$ *and* $f \in W^{2,\infty}(\lambda_1, \lambda_2; L^2(0, L))$ *be given. Then there exist* $\gamma > 0$ *such that for* $0 < \varepsilon < 1$*,*

$$\begin{aligned} &|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \\ &\le \gamma\Big(|v_1|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 + \varepsilon\Big). \end{aligned}$$

The proof follows the same general outline, but it is more involved. We again use Moreau-Yosida regularization (elasto-visco-plastic systems), but also their finite-dimensional (Galerkin) approximations, in order to obtain the a priori estimates needed to establish stability. Also, in the process of comparing dynamic solutions and the quasi-static solution we had to consider an auxiliary special dynamic solution as an intermediate step.

4 A generalization, applicable to 3-D

In the paper

A. Mielke, A. Petrov and J.A.C. Martins, Convergence of solutions of kinetic variational inequalities in the rate-independent quasi-static limit, J. Math. Anal. Appl. 348 (2008) 1012–1020,

the reader may find a more abstract and general treatment of the stability results mentioned in the previous Sections. Their study concerns evolutionary variational inequalities or inclusions of the form

$$M\ddot{q}(t) + Aq(t) + \partial\mathcal{R}(\dot{q}(t)) \ni l(t),$$

for the dynamics, and

$$A\bar{q}(\tau) + \partial\mathcal{R}(\bar{q}\,'(\tau)) \ni \bar{l}(\tau),$$

as the limit rate-independent system (l is the external loading and $\mathcal{R}$ the dissipation functional).

Instead of Moreau-Yosida regularization and time-differentiation, the authors rely on a difference quotient technique to obtain relatively simple and explicit bounds and then proceed to prove that kinetic evolutions remain close to a rate-independent path, if they start sufficiently close to the latter and if the load is applied sufficiently slowly, i.e. they prove stability in the sense of J.A.C. Martins et al.

The paper also contains an application to a 2-D or 3-D elastic plastic body with linear kinematic hardening.