On the equilibrium problem for masonry beams

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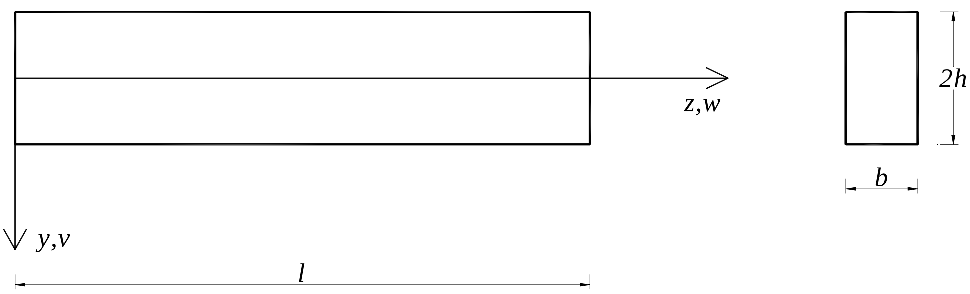


Rectilinear beam with rectangular section

$I = (0, l)$ reference configuration $\qquad z \in I$, material point

$\boldsymbol{u} = (w, v) \in W := W^{1,2} \times W^{2,2}$ $\qquad$ displacement field

$\widehat{\boldsymbol{e}}(\boldsymbol{u}) = (\epsilon, \kappa)$ $\qquad$ generalized strain

$$\epsilon = w' \qquad \kappa = -v''$$

(linearized) strain and change of curvature of the axis of the beam

$$N = \widehat{N} \circ \widehat{\boldsymbol{e}}(\boldsymbol{u}), \qquad M = \widehat{M} \circ \widehat{\boldsymbol{e}}(\boldsymbol{u}),$$

axial force and bending moment

$\boldsymbol{t} = (\widehat{N}, \widehat{M})$	generalized stresses
$V \subset W$	space of admissible displacements
$\boldsymbol{l} : V \to \mathbb{R}$	linear function of the loads
$< \boldsymbol{l}, \boldsymbol{u} >$	the work of the loads $\boldsymbol{l}$
$Y = L^2(I, \mathbb{R}^2)$	space of the generalized strains

We call $(V, \boldsymbol{l})$ external conditions.

We suppose that

$$\widehat{\boldsymbol{e}} : V \to Y$$

is injective and put

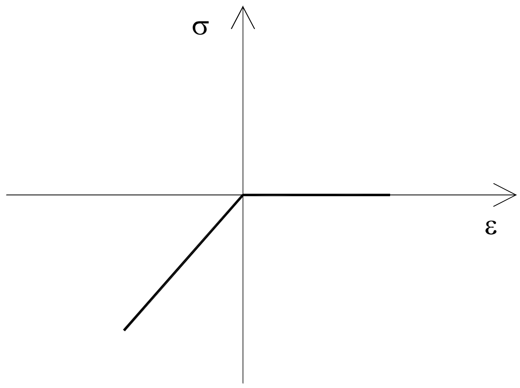
$$Y_0 = \left\{ \ \widehat{\boldsymbol{e}}(\boldsymbol{u}) \in Y : \boldsymbol{u} \in V \right\}.$$

We say that $\boldsymbol{t} \in Y$ equilibrates the loads if

$(\boldsymbol{t}, \ \widehat{\boldsymbol{e}}(\boldsymbol{u})) = \ < \boldsymbol{l}, \boldsymbol{u} >$, for each $\boldsymbol{u} \in V$.

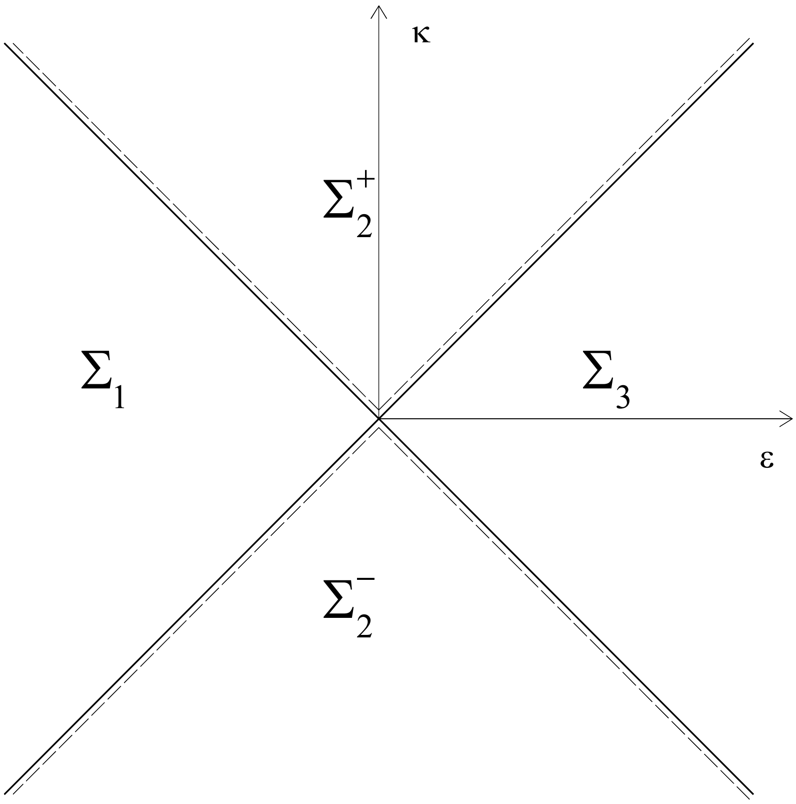
Proposition 1. If Y_0 is closed in Y, then there exists a stressfield $\boldsymbol{t}$ equilibrating the loads.

Constitutive equations



Starting from the general constitutive equation for (hyperelastic) masonry bodies [DEL PIERO, 1989], by employing Bernoulli-Euler hypothesis, we can obtain a constitutive equation for masonry beams.

Partition of the space of the generalized strains



$$\Sigma_1 = \left\{ \boldsymbol{e} = (\epsilon, \kappa) \in \mathbb{R}^2 : |\kappa| \leq \, -\epsilon \right\},$$

$$\Sigma_2^{\pm} = \left\{ \boldsymbol{e} = (\epsilon, \kappa) \in \mathbb{R}^2 : \, \pm\kappa > |\epsilon| \right\}$$

$$\Sigma_3 = \left\{ \boldsymbol{e} = (\epsilon, \kappa) \in \mathbb{R}^2 : \epsilon > 0, \, |\kappa| \leq \epsilon \right\},$$

The stored energy function

$\widehat{\phi} : \mathbb{R}^2 \to \mathbb{R}$,

$$\widehat{\phi}(\epsilon,\kappa) = \begin{cases} \dfrac{3\epsilon^2+\kappa^2}{6} & \text{if } (\epsilon,\kappa)\in\Sigma_1, \\ \dfrac{(\kappa\mp\epsilon)^3}{12\kappa} & \text{if } (\epsilon,\kappa)\in\Sigma_2^{\pm}, \\ 0 & \text{if } (\epsilon,\kappa)\in\Sigma_3, \end{cases}$$

Proposition 2. The function $\widehat{\phi}$ is convex, continuously differentiable and nonnegative with derivative $\mathrm{D}\widehat{\phi} = (\widehat{N},\widehat{M})$ given by

$$(\widehat{N},\widehat{M})(\epsilon,\kappa) = \begin{cases} (\epsilon, \kappa/3) & \text{if } (\epsilon,\kappa)\in\Sigma_1, \\ \mp\dfrac{(\kappa\mp\epsilon)^2}{12\kappa^2}(3\kappa,\ \mp 2\kappa-\epsilon) & \text{if } (\epsilon,\kappa)\in\Sigma_2^{\pm}, \\ \mathbf{0} & \text{if } (\epsilon,\kappa)\in\Sigma_3, \end{cases}$$

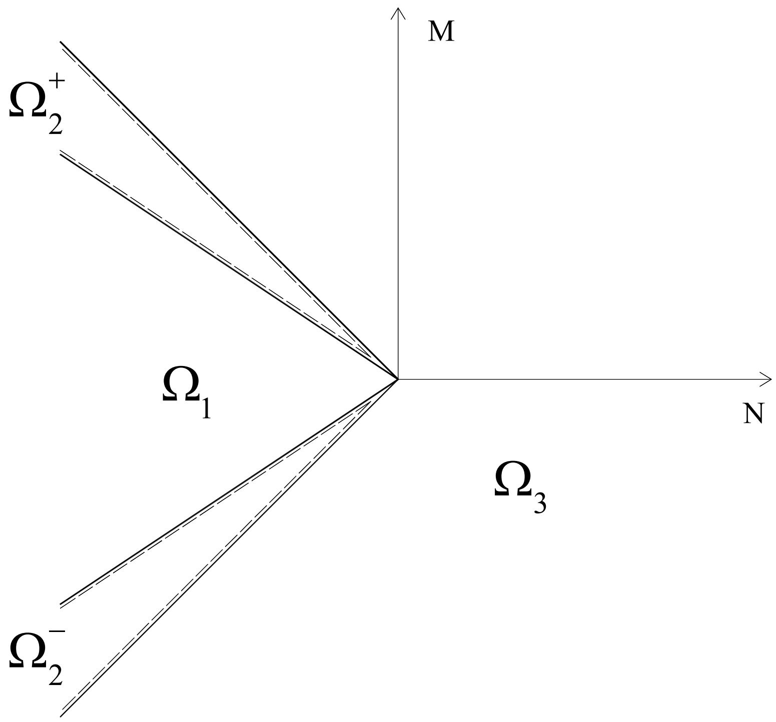
Moreover

$$\mathrm{D}\widehat{\phi}(\Sigma_1) = \Omega_1,$$

$$\mathrm{D}\widehat{\phi}(\Sigma_2^{\pm}) = \Omega_2^{\pm},$$

$$\mathrm{D}\widehat{\phi}(\Sigma_3) = \{\mathbf{0}\}.$$

Partition of the space of the generalized stresses



$$\Omega_1 = \left\{ \boldsymbol{t} = (N, M) : N \le 0,\, |M| \le\, -N/3 \right\},$$

$$\Omega_2^{\pm} = \left\{ \boldsymbol{t} = (N, M) : N < 0,\, |N|/3 <\, \pm M < |N| \right\},$$

$$\Omega_3 = \left\{ \boldsymbol{t} = (N, M) \ne \mathbf{0} : |M| \ge\, -N \right\};$$

We put

$$\Omega = \Omega_1 \cup \Omega_2^- \cup \Omega_2^+$$

Moreover we have

(i) $\widehat{\phi}(\epsilon,\kappa) \leq c|(\epsilon,\kappa)|^2, \quad c \in \mathbb{R}, \qquad (\epsilon,\kappa) \in \mathbb{R}^2;$

(ii) $\mathrm{D}\widehat{\phi}(\epsilon,\kappa) \leq c|(\epsilon,\kappa)|;$

(iii) $\widehat{\phi}$ is strictly convex in stresses, that is

$$\Big(\mathrm{D}\,\widehat{\phi}(\boldsymbol{e}_1) - \mathrm{D}\,\widehat{\phi}(\boldsymbol{e}_2)\Big) \cdot (\boldsymbol{e}_1 - \boldsymbol{e}_2) \geq c\Big|\mathrm{D}\,\widehat{\phi}(\boldsymbol{e}_1) - \mathrm{D}\,\widehat{\phi}(\boldsymbol{e}_2)\Big|^2$$

for some $c > 0$ and all $\boldsymbol{e}_1,\ \boldsymbol{e}_2 \in \mathbb{R}^2$.

Admissibility of the generalized stresses

We define

$\mathrm{H} : Y \to \mathbb{R}$,

$\mathrm{H}(\boldsymbol{e}) = \int_I \widehat{\phi} \circ \boldsymbol{e}\, dz,$

$\mathrm{H}^*(\boldsymbol{t}) :\ = \sup \left\{ \ (\boldsymbol{t}, \boldsymbol{a}) - \mathrm{H}(\boldsymbol{a}) : \boldsymbol{a} \in Y \right\} =$

$$\int_I \widehat{\phi}^* \circ \boldsymbol{t}\, dz,$$

where

$$\widehat{\phi}^*(\boldsymbol{s}) = \sup \left\{ \ (\boldsymbol{s} \cdot \boldsymbol{a}) - \widehat{\phi}(\boldsymbol{a}) : \boldsymbol{a} \in \mathbb{R}^2 \right\};$$

$$\widehat{\phi}^*(N, M) = \begin{cases} \dfrac{N^2 + 3M^2}{2} & \text{if } (N, M) \in \Omega_1, \\ \dfrac{4N^3}{9(N \pm M)} & \text{if } (N, M) \in \Omega_2^{\pm}, \\ \infty & \text{if } (N, M) \in \Omega_3. \end{cases}$$

We say that $\boldsymbol{t}$ is *admissible* if $\mathrm{H}^*(\boldsymbol{t}) < \infty$.

Let $\sigma : \mathbb{R}^2 \to [0,\infty]$,

$$\sigma(\boldsymbol{t}) = \begin{cases} 0 & \text{if } \boldsymbol{t} = \mathbf{0}, \\ \dfrac{|N|^3}{|N| - |M|} & \text{if } \boldsymbol{t} = (N,M) \in \Omega,\ \boldsymbol{t} \neq \mathbf{0}, \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 3. For a no-tension beam the external conditions $(V, \boldsymbol{l})$ are compatible if and only if there exists a stressfield $\boldsymbol{t} \in Y$ equilibrating the loads such that

$$\int_I \sigma \circ \boldsymbol{t}\, dz < \infty.$$

Roughly, this last condition is equivalent to say that the *line of the thrust* is wholly contained within the (open) longitudinal section of the beam.

Infimum energy

$$\mathrm{F}(\boldsymbol{u}) = \int_I \widehat{\phi} \circ \widehat{\boldsymbol{e}}(\boldsymbol{u})\, dz \qquad \text{internal energy}$$

$$\mathrm{E}(\boldsymbol{u}) = \mathrm{F}(\boldsymbol{u}) - <\boldsymbol{l}, \boldsymbol{u}> \qquad \text{potential energy}$$

$$\mathrm{E}_0 = \inf \big\{ \mathrm{E}(\boldsymbol{u}) : \boldsymbol{u} \in V \big\} \in \mathbb{R} \cup \big\{ -\infty \big\}$$

is the *infimum energy*.

Proposition 4. $\mathrm{E}_0 > -\infty$ if and only if there exists an admissible stressfield $\boldsymbol{t}$ equilibrating the loads. If these conditions are satisfied we have

$$\mathrm{E}_0 = -\min \big\{ \mathrm{H}^*(\boldsymbol{t}) : \boldsymbol{t} \in Y, \boldsymbol{t} \text{ equilibrates the loads} \big\}.$$

Proposition 5. (i) If there exist an admissible generalized strainfield $\boldsymbol{e} \in Y_0$ and a stressfield $\boldsymbol{t} \in Y$ equilibrating the loads, such that $\mathrm{D}\,\widehat{\phi}(\boldsymbol{e}) = \boldsymbol{t}$, then the external conditions have an equilibrium state and $\boldsymbol{t}$ is admissible.

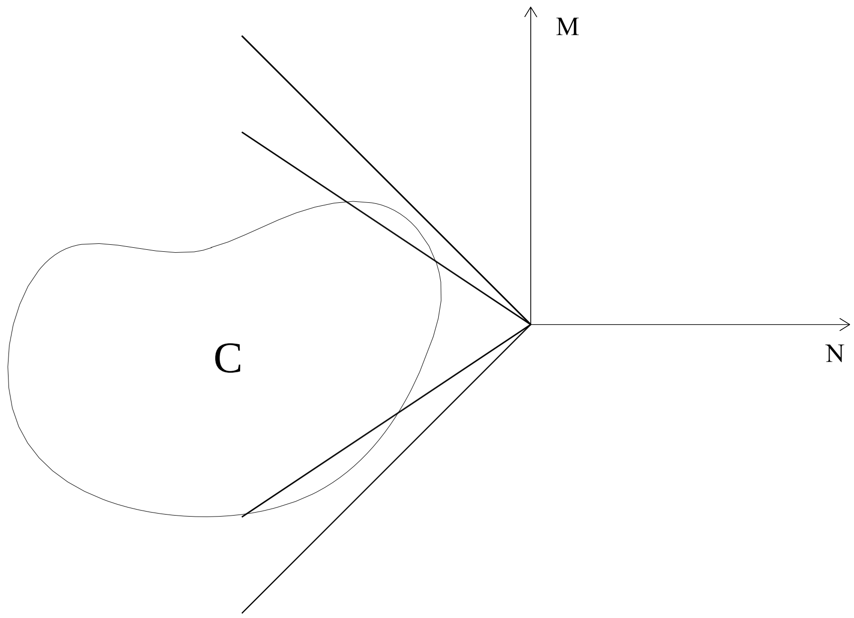
Conversely, if $\boldsymbol{u}_0$ is an equilibrium state, than $\boldsymbol{t} = \mathrm{D}\,\widehat{\phi}(\boldsymbol{e}_0) \in Y$ is an admissible stressfield that equilibrates the loads.

(ii) If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two equilibrium states and $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ the stressfields corresponding to them, then $\boldsymbol{t}_1 = \boldsymbol{t}_2$.

If $(N, M) \in \Omega \backslash \{0\}$, the constitutive equation

$$(N, M) = (\widehat{N}, \widehat{M})(\epsilon, \kappa)$$

can be inverted. Thus, if there exists a closed set $C \subset \Omega \backslash \{0\}$ such that the essential range of the (unique) stressfield corresponding to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is contained in C, then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.



Example

$$N = -F, \qquad M = \lambda(z - l).$$

The collapse load is $\lambda_c = Fh/l$.

For $\lambda \leq \frac{Fh}{3l}$, we have the elastic solution $\boldsymbol{u} = (w, v)$,

$$w = -\frac{Fz}{2Ebh}, \qquad v = \frac{\lambda z^2(3l - z)}{4Ebh^3},$$

$$\widehat{\phi} = \frac{3\lambda^2 z^2 - 6l\lambda^2 z + F^2h^2 + 3l^2\lambda^2}{4Ebh^3},$$

and then

$$\mathrm{F}(\boldsymbol{u}) = \int_I \widehat{\phi} \circ \widehat{\boldsymbol{e}}(\boldsymbol{u})\, dz = \frac{l(F^2h^2 + l^2\lambda^2)}{4Ebh^3}.$$

The work of the loads is

$$< \boldsymbol{l}, \boldsymbol{u} > \; = \frac{F^2 l}{2Ebh} + \frac{\lambda^2 l^3}{2Ebh^3}$$

and then the potential energy is

$$\mathrm{E}(\boldsymbol{u}) = -\frac{l(F^2h^2 + l^2\lambda^2)}{4Ebh^3}.$$

For $\dfrac{Fh}{3l} \leq \lambda \leq \dfrac{Fh}{l}$, we have

$$\widehat{\phi} = \begin{cases} \dfrac{2F^3}{9Eb(\lambda z + Fh - l\lambda)}, & 0 \leq z \leq z_0, \\[2ex] \dfrac{3\lambda^2 z^2 - 6l\lambda^2 z + F^2h^2 + 3l^2\lambda^2}{4Ebh^3}, & z_0 < z \leq l \end{cases}$$

with $z_0 = l - \dfrac{Fh}{3\lambda}$.

Thus

$$\mathrm{F}(\boldsymbol{u}) = \int_0^{z_0} \frac{2F^3}{9Eb(\lambda z + Fh - l\lambda)} dz +$$

$$\int_0^{z_0} \frac{3\lambda^2 z^2 - 6l\lambda^2 z + F^2h^2 + 3l^2\lambda^2}{4Ebh^3} dz \; =$$

$$\frac{2F^3 ln(\frac{3}{2}Fh(Fh-l\lambda))}{9Eb\lambda} + \frac{5F^3}{54Eb\lambda}.$$

In the same way we can obtain the displacement field $\boldsymbol{u}$ and the work of the loads, from which we get the potential energy

$$\mathrm{E}(\boldsymbol{u}) = \frac{2F^3 ln\big(\frac{3}{2}Fh(Fh-l\lambda)\big)}{9Eb\lambda} - \frac{5F^3}{54Eb\lambda}.$$