Energetic Consistency Conditions for Perfect Collisions

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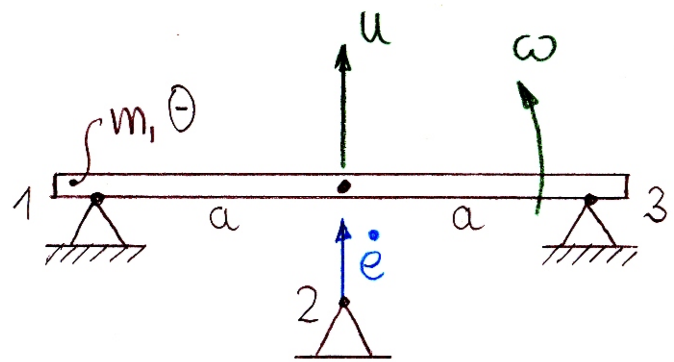
For given $\vec{u}^-$, $\vec{u}^+$ at a frictionless collision, decide whether the contacts supplied the system with energy or not.



2 degrees of freedom

3 contacts

⇝ <u>not</u> independent constraints

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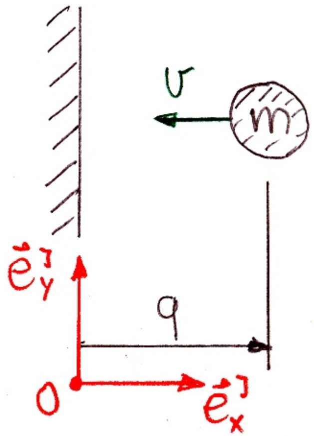
1. Problem

$\vec{q}$: gen. coordinates, $\vec{u}$: gen. velocities ($\dot{\vec{q}} = \vec{u}$ a.e.)

Impact against fixed wall



inelastic impact:

$u^- = -v$

$u^+ = 0$

- Kinetic energy

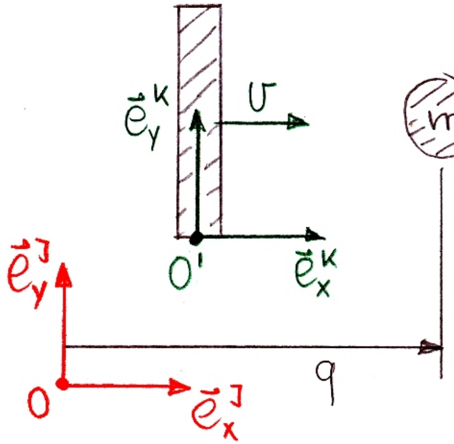
  $T^- = \frac{1}{2} m v^2 \qquad T^+ = 0$

- Dissipation if

  $T^+ < T^-$

  $T^+ - T^- = 0 - \frac{1}{2} m v^2 < 0$ ok

Impact against moving wall



inelastic impact:

$u^- = 0$

$u^+ = v$

- Energy difference (red)

  $T^+ - T^- = \frac{1}{2} m v^2 - 0 > 0$ bad!

- Energy difference (green)

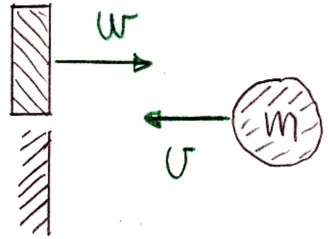
  $T^+ - T^- = 0 - \frac{1}{2} m v^2 < 0$ good!

- Solution

  Find transformation

  moving walls → fixed walls

Too many moving walls



one dof, two constraints

- Assumption:

  Ball hits simultaneously moving and fixed wall

- Problem

  Transformation to fixed walls no longer exists!

  ! no pathological case

2. First Conclusions

Absolute Velocities:

- used for kinetic energy
- linear affine space
- artificial magnitudes
- T not frame-invariant

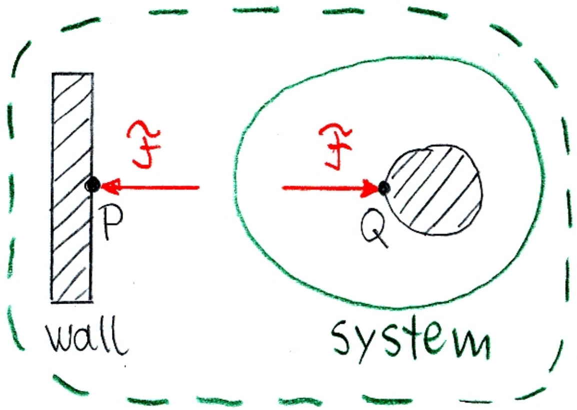
Relative Velocities:

- space of "difference velocities"
- linear space with natural 0 element
- not used for kinetic energy

$\Rightarrow$ find another appropriate measure!

Strategy:

- find dissipation measure based on relative velocities
- internal forces occur as pairs
- add walls as bodies with infinite mass to system



$\rightarrow$ **Contact Work**

$$W = \int_{\{t\}} \vec{v}_P^T \, d\vec{F} - \vec{v}_Q^T \, d\vec{F} = \int_{\{t\}} (\vec{v}_P - \vec{v}_Q)^T \, d\vec{F}$$

3. The Contact Work for Impacts

Original System:

$$M(\vec{u}^+-\vec{u}^-)=\vec{w}\,\Lambda$$

$$\gamma^\pm=\vec{w}^T\vec{u}^\pm+\underbrace{\chi}_{\text{kinematic excitation}}$$

Extended System:

$$\begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \vec{u}^+-\vec{u}^- \\ \chi^+-\chi^- \end{pmatrix}=\begin{pmatrix} \vec{w} \\ 1 \end{pmatrix}\Lambda$$

$$\delta^\pm=\begin{pmatrix} \vec{w} \\ 1 \end{pmatrix}^T\begin{pmatrix} \vec{u}^\pm \\ \chi^\pm \end{pmatrix}$$

Proposition: Systems are equivalent for $m\to\infty$, $\chi^+\to\chi^-$:

$$\underset{\downarrow\,\infty}{m}\,\underbrace{(\chi^+-\chi^-)}_{\downarrow\,0}=\Lambda \quad \text{finite}$$

$$\Rightarrow \underline{\underline{\delta\to\gamma}}$$

Energy Difference (autonomous system!)

$$T^+-T^-=\frac{1}{2}\begin{pmatrix} \vec{u}^+ \\ \chi^+ \end{pmatrix}^T\begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \vec{u}^+ \\ \chi^+ \end{pmatrix}-\frac{1}{2}\begin{pmatrix} \vec{u}^- \\ \chi^- \end{pmatrix}^T\begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \vec{u}^- \\ \chi^- \end{pmatrix}$$

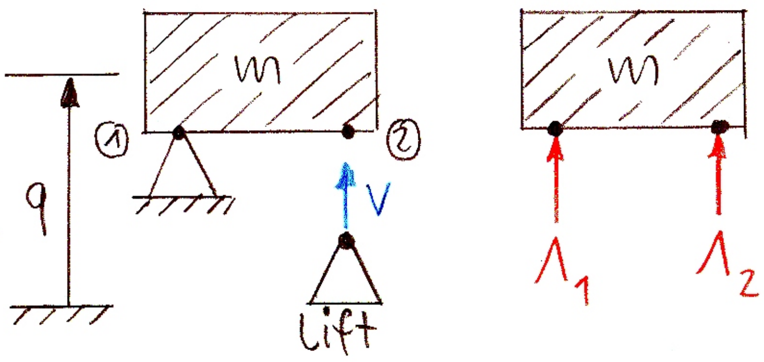
$$=\frac{1}{2}\begin{pmatrix} \vec{u}^++\vec{u}^- \\ \chi^++\chi^- \end{pmatrix}^T\begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \vec{u}^+-\vec{u}^- \\ \chi^+-\chi^- \end{pmatrix}=\frac{1}{2}\begin{pmatrix} \vec{u}^++\vec{u}^- \\ \chi^++\chi^- \end{pmatrix}^T\begin{pmatrix} \vec{w} \\ 1 \end{pmatrix}\Lambda$$

$$=\frac{1}{2}\vec{w}^T(\vec{u}^++\vec{u}^-)\Lambda+\frac{1}{2}(\chi^++\chi^-)\Lambda=\frac{1}{2}(\vec{w}^T\vec{u}^++\chi^+)\Lambda+\frac{1}{2}(\vec{w}^T\vec{u}^-+\chi^-)\Lambda$$

$$=\frac{1}{2}(\delta^++\delta^-)\Lambda\xrightarrow{m\to\infty}\underline{\underline{\frac{1}{2}(\gamma^++\gamma^-)\Lambda=:W}}\quad \text{Contact Work}$$

4. Example: Non-Uniform Kinematic Excitation

Mass resting on the ground and waiting for a lift



pre-, post-impact velocity

$$u^- := 0 \qquad u^+ := v$$

relative velocities

$$\gamma_1 = u \qquad \gamma_2 = u - v$$

$$\Rightarrow \gamma_1^- = 0 \quad \gamma_1^+ = v \quad \gamma_2^- = -v \quad \gamma_2^+ = 0$$

kinetics (impact equations)

$$m(u^+ - u^-) = \Lambda_1 + \Lambda_2 \quad \Rightarrow \quad mv = \Lambda_1 + \Lambda_2 \quad (\Lambda_i \geq 0)$$

contact work

$$W = \frac{1}{2}(\gamma_1^+ + \gamma_1^-)\Lambda_1 + \frac{1}{2}(\gamma_2^+ + \gamma_2^-)\Lambda_2$$

$$= \frac{1}{2}v\Lambda_1 - \frac{1}{2}v\Lambda_2 = \frac{1}{2}v(\Lambda_1 - \Lambda_2)$$

from kinetics

$$\Lambda_2 = mv - \Lambda_1 \quad ; \quad 0 \leq \Lambda_1 \leq mv$$

in contact work

$$W = \frac{1}{2}v(\Lambda_1 - mv + \Lambda_1)$$

$$= -\frac{1}{2}v^2 m + v\Lambda_1 \quad \text{with } 0 \leq \Lambda_1 \leq mv$$

$$\Rightarrow W \in \left[-\frac{1}{2}mv^2, +\frac{1}{2}mv^2\right]$$ interval!

5. Problem Setting ($\dot{\vec q} = \vec u$ a.e.)

- Admissible Positions at t

$$\mathcal{C} := \{\vec q \mid g_i(\vec q,t) \geq 0 \ \forall i\}$$

- Active Constraints at t

$$\mathcal{H}(\vec q,t) := \{i \mid g_i(\vec q,t) = 0\}$$

- Kinematics

$\gamma_i(\vec u) = \vec w_i^T(\vec q,t)\,\vec u + \chi_i(\vec q,t)$ rel. velocities

$\mathcal{T}_{\mathcal{C}}(\vec q,t) = \{\vec u \mid \vec w_i^T \vec u \geq 0 \ \forall i\}$

$\mathcal{P}^+_{\mathcal{C}}(\vec q,t) = \{\vec u \mid \gamma_i(\vec u) \geq 0 \ \forall i\}$; $\vec u^+ \in \mathcal{P}^+_{\mathcal{C}}$

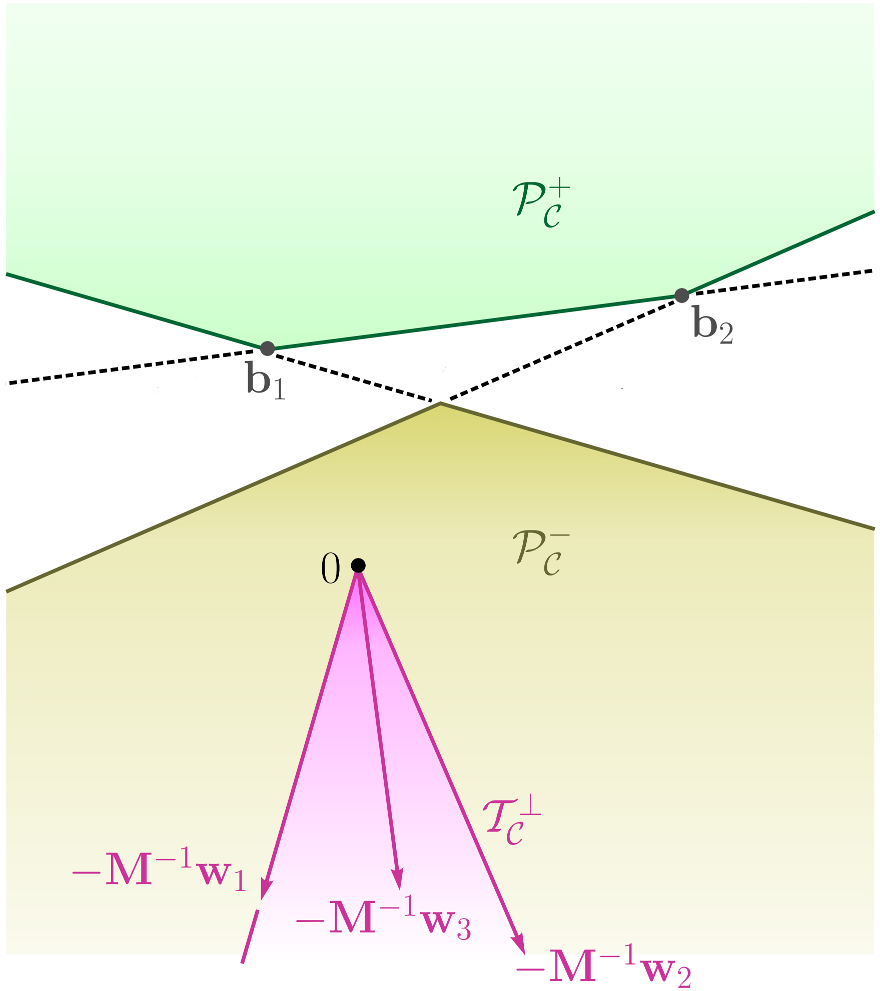
$\mathcal{P}^-_{\mathcal{C}}(\vec q,t) = \{\vec u \mid \gamma_i(\vec u) \leq 0 \ \forall i\}$; $\vec u^- \in \mathcal{P}^-_{\mathcal{C}}$

- Kinetics

$M(\vec q)(\vec u^+ - \vec u^-) = \sum_{i\in\mathcal{H}} \vec w_i(\vec q,t)\,\Lambda_i$; $\Lambda_i \geq 0$

$M(\vec q)(\vec u^+ - \vec u^-) \in -\mathcal{N}_{\mathcal{C}}(\vec q,t)$

$(\vec u^+ - \vec u^-) \in -\mathcal{T}^\perp_{\mathcal{C}}(\vec q,t)$



6. Representation of the Contact Work

$\gamma_i^+ := \gamma_i(\vec{u}^+)$ $\quad \gamma_i^- := \gamma_i(\vec{u}^-)$ $\quad \gamma_i(\vec{u}) = \vec{w}_i^T \vec{u} + \chi_i$

$$W = \sum \tfrac{1}{2}(\gamma_i^+ + \gamma_i^-)\, \Lambda_i$$

$$= \sum \tfrac{1}{2}\left(\vec{w}_i^T(\vec{u}^+ + \vec{u}^-) + 2\chi_i\right) \Lambda_i$$

$$= \sum \tfrac{1}{2}\left(\vec{w}_i^T(\vec{u}^+ - \vec{b} + \vec{u}^- - \vec{b}) + 2(\vec{w}_i^T \vec{b} + \chi_i)\right) \Lambda_i$$

Note: W is <u>independent</u> of $\vec{b}$!

$$= \tfrac{1}{2}\sum (\vec{u}^+ - \vec{b} + \vec{u}^- - \vec{b})^T \vec{w}_i \Lambda_i + \sum (\vec{w}_i^T \vec{b} + \chi_i) \Lambda_i$$

<u>Kinetics:</u> $\sum \vec{w}_i \Lambda_i = M(\vec{u}^+ - \vec{u}^-) = M\left((\vec{u}^+ - \vec{b}) - (\vec{u}^- - \vec{b})\right)$

$$= \tfrac{1}{2}\left((\vec{u}^+ - \vec{b}) + (\vec{u}^- - \vec{b})\right)^T M \left((\vec{u}^+ - \vec{b}) - (\vec{u}^- - \vec{b})\right) + \sum (\vec{w}_i^T \vec{b} + \chi_i) \Lambda_i$$

$$= \tfrac{1}{2}\|\vec{u}^+ - \vec{b}\|_M^2 - \tfrac{1}{2}\|\vec{u}^- - \vec{b}\|_M^2 + \sum (\vec{w}_i^T \vec{b} + \chi_i) \Lambda_i$$

$$\Rightarrow W(\vec{b}, \Lambda_i) = \underbrace{\tfrac{1}{2}\|\vec{u}^+ - \vec{b}\|_M^2 - \tfrac{1}{2}\|\vec{u}^- - \vec{b}\|_M^2}_{\equiv -(\vec{u}^+ - \vec{u}^-)^T M \vec{b} + \text{const.}} + \sum \underbrace{(\vec{w}_i^T \vec{b} + \chi_i)}_{\gamma_i(\vec{b})} \underset{\geq 0}{\Lambda_i}$$

still <u>independent</u> of $\vec{b}$!

7. Energetic Consistency via Contact Work

$$W(\vec{b},\Lambda_i) = \underbrace{\tfrac{1}{2}\|\vec{u}^+-\vec{b}\|_M^2 - \tfrac{1}{2}\|\vec{u}^--\vec{b}\|_M^2}_{\equiv -(\vec{u}^+-\vec{u}^-)^T M\vec{b} + \text{const.}} + \sum \underbrace{(\vec{w}_i^T\vec{b}+\chi_i)}_{\gamma_i(\vec{b})} \Lambda_i$$

$\Lambda_i \geq 0$

Saddle-function $\leftrightarrow$ Lagrangian!

Proposition: Let $\vec{u}^+, \vec{u}^-$ be given and let $\Lambda_i \geq 0$ such that $M(\vec{u}^+-\vec{u}^-) = \sum \vec{w}_i \Lambda_i$.
Then there exists a tuple $\Lambda_{i0} \geq 0$ and a $\vec{b}_0 \in \mathcal{P}_e^+$ such that

(1) $M(\vec{u}^+-\vec{u}^-) = \sum \vec{w}_i \Lambda_{i0}$

(2) $\sum \gamma_i(\vec{b}_0)\Lambda_{i0} = 0 \qquad (0 \leq \Lambda_{i0} \perp \gamma_i(\vec{b}_0) \geq 0)$

(3) $\sum \gamma_i(\vec{b}_0)\Lambda_i \geq 0 \qquad (\gamma_i(\vec{b}_0) \geq 0;\ \Lambda_i \geq 0)$

(4) $\tfrac{1}{2}\|\vec{u}^+-\vec{b}_0\|_M^2 - \tfrac{1}{2}\|\vec{u}^--\vec{b}_0\|_M^2 \geq$
$\tfrac{1}{2}\|\vec{u}^+-\vec{b}\|_M^2 - \tfrac{1}{2}\|\vec{u}^--\vec{b}\|_M^2 \quad \forall \vec{b} \in \mathcal{P}_e^+$

Consequences on Contact Work:

$$W(\vec{b}_0,\Lambda_{i0}) = \tfrac{1}{2}\|\vec{u}^+-\vec{b}_0\|_M^2 - \tfrac{1}{2}\|\vec{u}^--\vec{b}_0\|_M^2 + 0 \qquad (2)$$

$$W(\vec{b}_0,\Lambda_i) = \tfrac{1}{2}\|\vec{u}^+-\vec{b}_0\|_M^2 - \tfrac{1}{2}\|\vec{u}^--\vec{b}_0\|_M^2 + c \quad (c \geq 0) \quad (3)$$

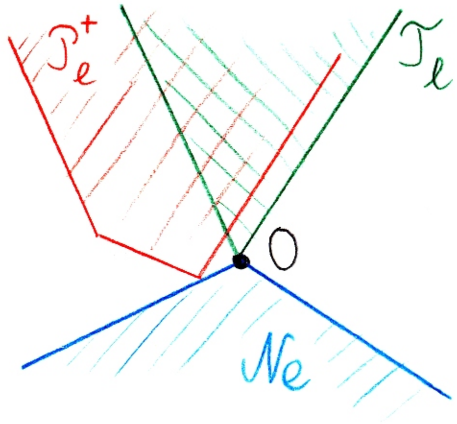
$\Rightarrow \underline{\underline{W(\vec{b}_0,\Lambda_{i0}) \leq W(\vec{b}_0,\Lambda_i)}}$ $\forall \Lambda_i$ leading to the same $\vec{u}^+$

$\Rightarrow$ We know now how to determine for a given $\vec{u}^+, \vec{u}^-$ the distribution Λ_{i0} of the contact impulses that leads to the <u>minimal possible contact work</u> $W(\vec{b}_0,\Lambda_{i0})$

(same can be done for maximal contact work)

<u>Proof:</u> With $\vec{b}_0$ as the optimal solution, (1)-(4) follows from the linear opt. problem

$$\text{Minimize } f(\vec{b}) = (\vec{u}^+ - \vec{u}^-)^T M \vec{b} \text{ under } \vec{b} \in \mathcal{P}_e^+$$



<u>a) Existence of solution $\vec{b}_0$:</u> $\mathcal{T}_e$ is recession cone of $\mathcal{P}_e^+$.

With $\vec{b} \in \mathcal{T}_e$ and $M(\vec{u}^+ - \vec{u}^-) \in -\mathcal{N}_e$ it follows that

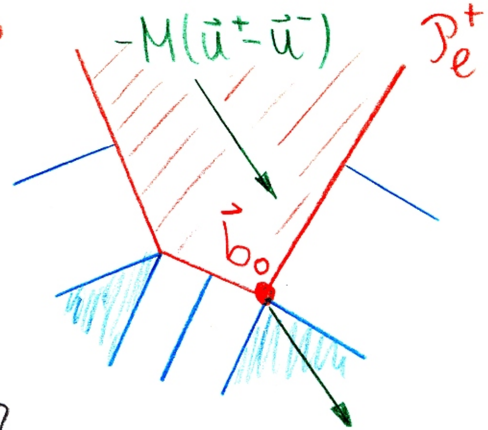
$(\vec{u}^+ - \vec{u}^-)^T M \vec{b} \geq 0$, i.e. f is bounded from below. $\rightsquigarrow$ Min. exists

<u>b) Minimizer $\vec{b}_0$ of $f(\vec{b})$ on $\mathcal{P}_e^+$:</u>

$(\vec{u}^+ - \vec{u}^-)^T M \vec{b}_0 \leq (\vec{u}^+ - \vec{u}^-)^T M \vec{b}$ with $\vec{b}_0 \in \mathcal{P}_e^+$, $\forall \vec{b} \in \mathcal{P}_e^+$ (4) ✓

$\Leftrightarrow -M(\vec{u}^+ - \vec{u}^-) \in \mathcal{N}_{\mathcal{P}_e^+}(\vec{b}_0) \subset \mathcal{N}_e$

! "inverse" normal cone problem; finite number of corner points



<u>c) Representation of normal cone:</u>

$-M(\vec{u}^+ - \vec{u}^-) \in \mathcal{N}_{\mathcal{P}_e^+}(\vec{b}_0)$ with $\mathcal{P}_e^+ = \{\vec{b} \mid \gamma_i(\vec{b}) = \vec{w}_i^T \vec{b} + \chi_i \geq 0\}$?

$$\Leftrightarrow \underbrace{-M(\vec{u}^+ - \vec{u}^-) = -\sum \Lambda_{i0} \vec{w}_i}_{(1)\ ✓}, \quad \gamma_i(\vec{b}_0) \geq 0, \quad \Lambda_{i0} \geq 0, \quad \underbrace{\gamma_i(\vec{b}_0) \Lambda_{i0} = 0}_{(2)\ ✓}$$

• Minimal contact work for given $\vec{u}^+, \vec{u}^-$ $\qquad$ $\mathcal{E}$: set of corner points of $\mathcal{P}_e^+$

$$W(\vec{b}_0, \Lambda_{i0}) = \tfrac{1}{2}\|\vec{u}^+ - \vec{b}_0\|_M^2 - \tfrac{1}{2}\|\vec{u}^- - \vec{b}_0\|_M^2 =: \alpha \iff \vec{u}^+ \in \partial\mathcal{B}_{\sqrt{2\alpha + \|\vec{u}^- - \vec{b}_0\|_M^2}}(\vec{b}_0)$$

$$(4): \forall \vec{b} \in \mathcal{P}_e^+ : \tfrac{1}{2}\|\vec{u}^+ - \vec{b}\|_M^2 - \tfrac{1}{2}\|\vec{u}^- - \vec{b}\|_M^2 \leq \alpha \iff \vec{u}^+ \in \mathcal{B}_{\sqrt{2\alpha + \|\vec{u}^- - \vec{b}\|_M^2}}(\vec{b}) \ \forall \vec{b} \in \mathcal{P}_e^+$$

$$\Rightarrow \underline{\underline{\vec{u}^+ \in \bigcap_{\vec{b} \in \mathcal{E}} \mathcal{B}_{\sqrt{2\alpha + \|\vec{u}^- - \vec{b}\|_M^2}}(\vec{b})}}$$

• Conversely, if it holds for any $\vec{u}^+ \in \mathcal{P}_c^+ \cap (\vec{u}^- - \mathcal{J}_c^\perp)$ that

$$\tfrac{1}{2}\|\vec{u}^+ - \vec{b}\|_M^2 - \tfrac{1}{2}\|\vec{u}^- - \vec{b}\|_M^2 \leq \alpha \quad \forall \vec{b} \in \mathcal{E} \iff \vec{u}^+ \in \bigcap_{\vec{b} \in \mathcal{E}} \mathcal{B}_{\sqrt{2\alpha + \|\vec{u}^- - \vec{b}\|_M^2}}(\vec{b})$$

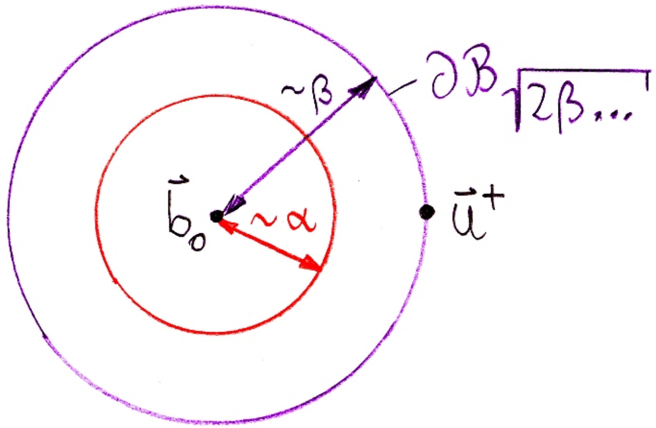
then $\underline{\underline{W(\vec{b}_0(\vec{u}^+), \Lambda_{i0}) \leq \alpha}}$

<u>Proof:</u> Let $W(\vec{b}_0(\vec{u}^+), \Lambda_{i0}) = \beta$.

$\Rightarrow \exists\, \vec{b}_0 \in \mathcal{E}$ such that $\vec{u}^+ \in \partial\mathcal{B}_{\sqrt{2\beta + \|\vec{u}^- - \vec{b}_0\|_M^2}}(\vec{b}_0)$

Assume $\beta > \alpha \Rightarrow \sqrt{2\beta + \|\vec{u}^- - \vec{b}_0\|_M^2} > \sqrt{2\alpha + \|\vec{u}^- - \vec{b}_0\|_M^2}$

$\Rightarrow \vec{u}^+ \notin \mathcal{B}_{\sqrt{2\alpha + \|\vec{u}^- - \vec{b}\|_M^2}}(\vec{b})$ for this $\vec{b} = \vec{b}_0 \in \mathcal{E}$ ↯ $\Rightarrow \underline{\underline{\beta \leq \alpha}}$



8. Construction of the Consistency Set for $\vec{u}^+$

- Kinematic Compatibility

  $\vec{u}^+ \in \mathcal{P}_e^+ \quad (\vec{u}^- \in \mathcal{P}_e^-)$

- Kinetic Compatibility

  $\vec{u}^+ \in \vec{u}^- - \mathcal{T}_e^\perp$

- Energetic Consistency

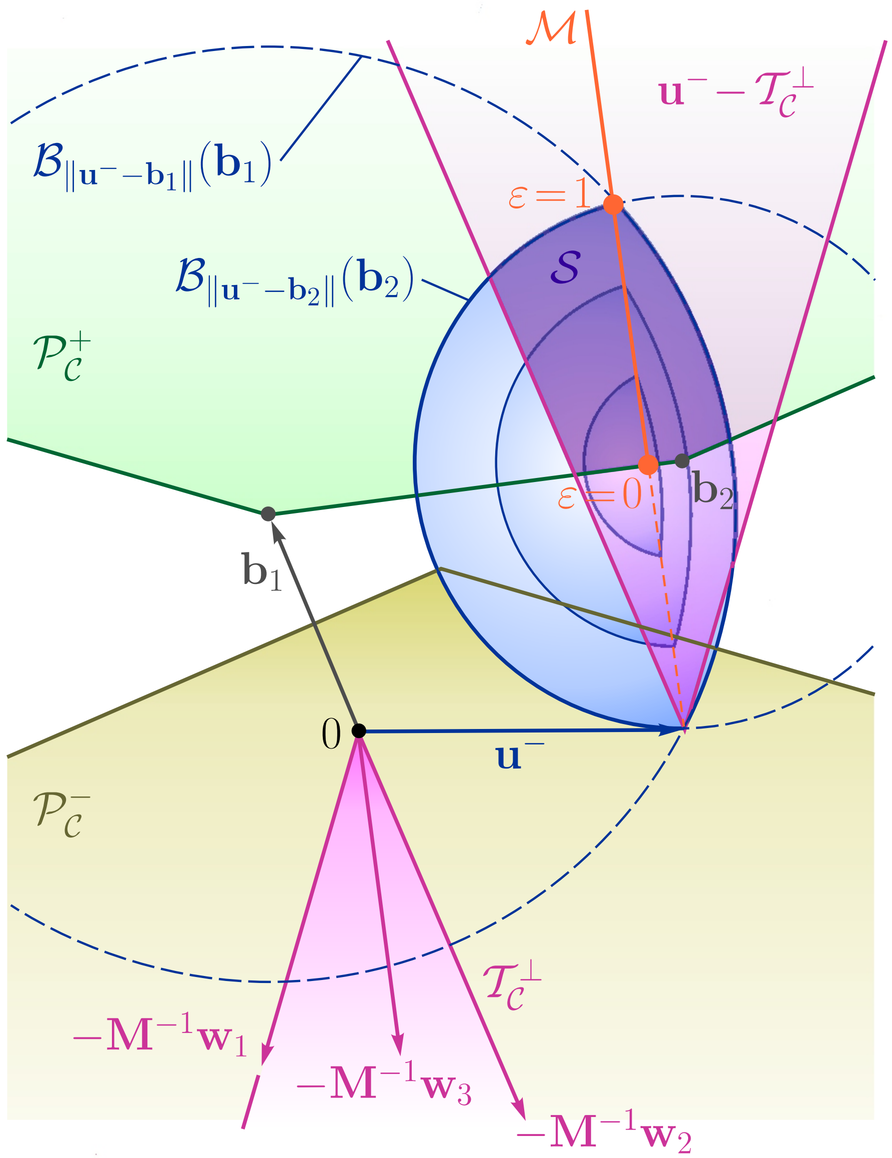
  $\vec{u}^+ \in \bigcap_{\vec{b} \in \mathcal{E}} \mathcal{B}_{\sqrt{2\alpha + \|\vec{u}^- - \vec{b}\|_M^2}}(\vec{b}) =: \mathcal{W}_\alpha$

  $\Updownarrow$

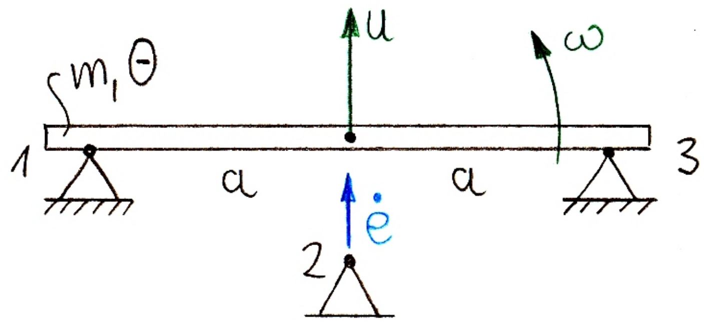
  contact work $\leq \alpha$

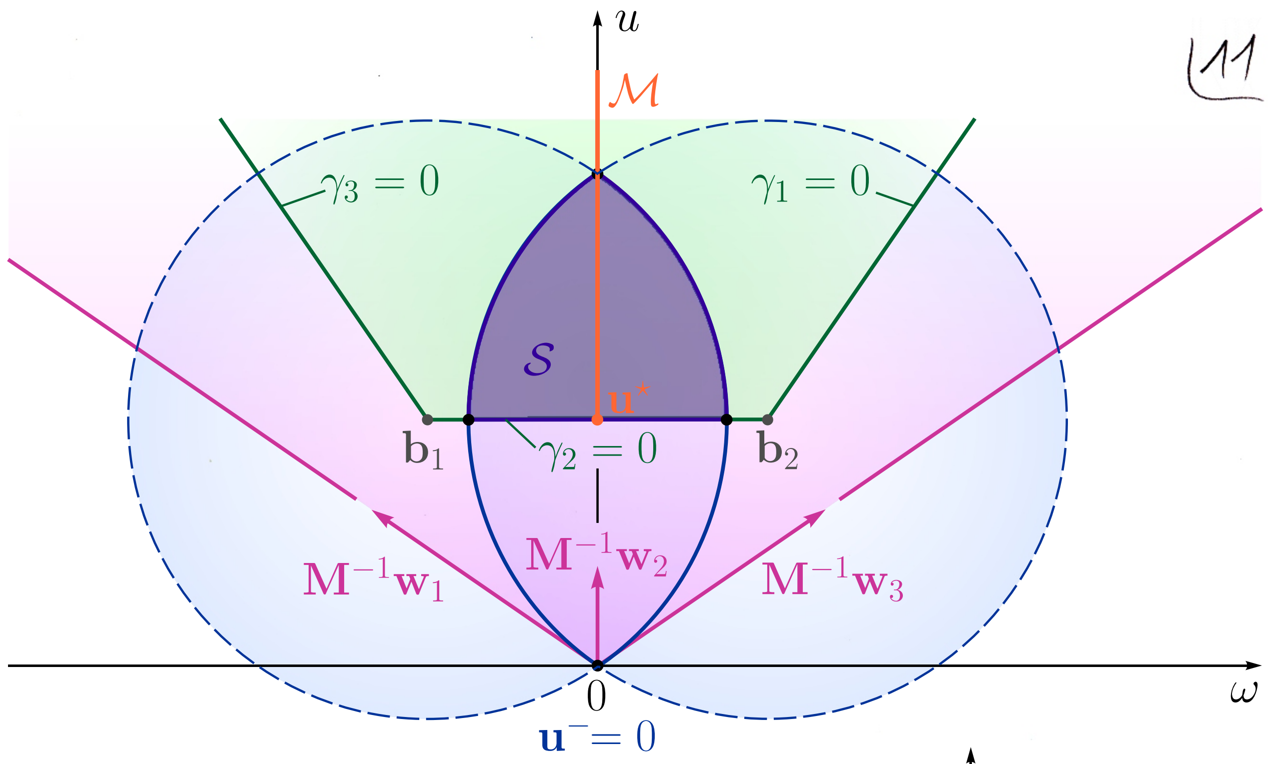
- Consistency Set $(\alpha = 0)$

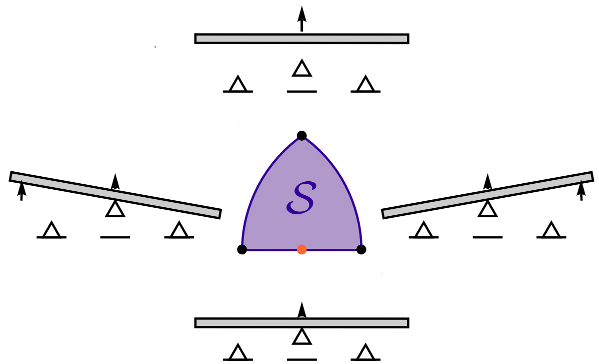
  $\mathcal{S} = \mathcal{P}_e^+ \cap (\vec{u}^- - \mathcal{T}_e^\perp) \cap \mathcal{W}_0$



- Example: Forklift truck







generalized velocities

$$\vec{u} = (u, \omega)^T$$

pre-impact velocity

$$\vec{u}^- = (0, 0)^T$$

relative velocities

$$\gamma_1 = u - a\omega \qquad \gamma_2 = u - \dot{e} \qquad \gamma_3 = u + a\omega$$

generalized force directions

$$\vec{w}_1 = \begin{pmatrix} 1 \\ -a \end{pmatrix} \qquad \vec{w}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{w}_3 = \begin{pmatrix} 1 \\ a \end{pmatrix}$$

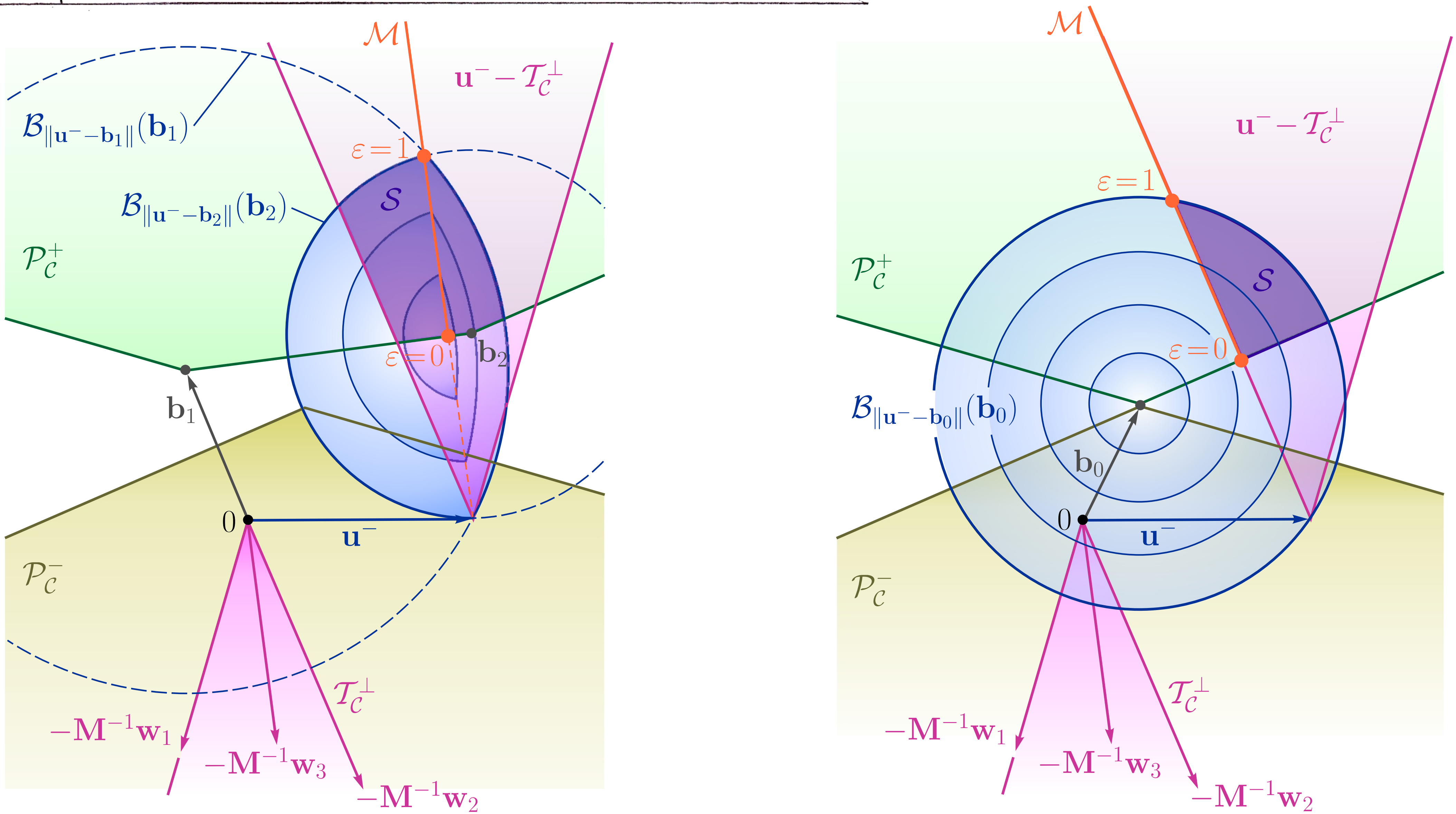
mass matrix

$$M = \begin{pmatrix} m & 0 \\ 0 & \Theta \end{pmatrix}$$

energetic consistency ($\alpha = 0$)

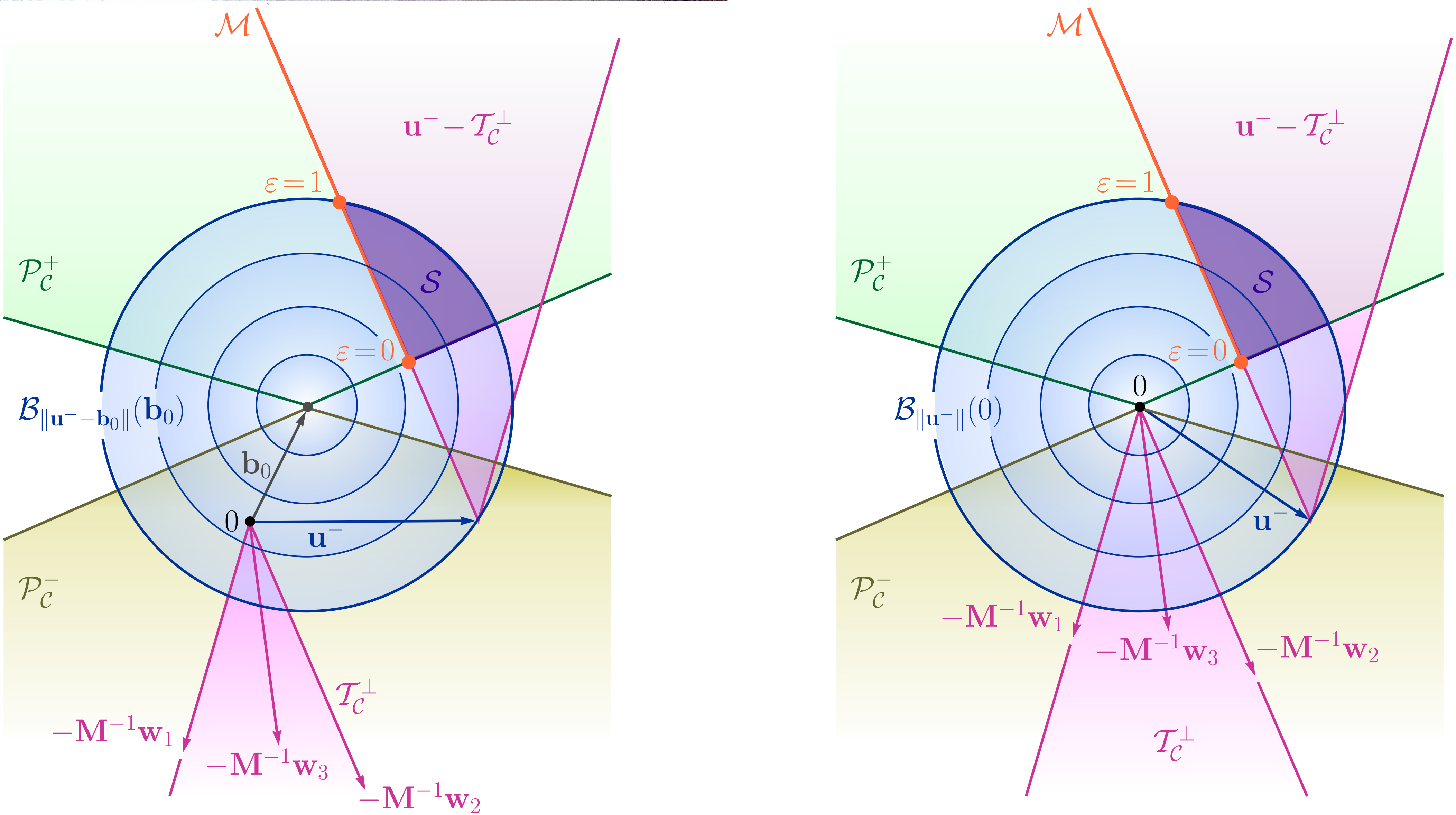
$$\vec{u}^+ \in \bigcap_{\vec{b} \in \mathcal{E}} \mathcal{B}_{\|\vec{b}\|_M}(\vec{b})$$

9. Special Case: Uniform Kinematic Excitation



$$\left.\begin{array}{l|l} W(\vec{b}_0, \Lambda_{i0}) = \tfrac{1}{2}\|\vec{u}^+ - \vec{b}_0\|_M^2 - \tfrac{1}{2}\|\vec{u}^- - \vec{b}_0\|_M^2 + 0 & c = \sum(\vec{w}_i^T \vec{b}_0 + \chi_i)\,\Lambda_i \geq 0 \\ W(\vec{b}_0, \Lambda_i) = \tfrac{1}{2}\|\vec{u}^+ - \vec{b}_0\|_M^2 - \tfrac{1}{2}\|\vec{u}^- - \vec{b}_0\|_M^2 + c & \text{Find } \vec{b}_0 \text{ s.t. } \vec{w}_i^T \vec{b}_0 + \chi_i = 0 \end{array}\right\} \Rightarrow \begin{array}{l} W \ \underline{\underline{\text{unique}}} \\ \gamma_i(\vec{u}) = \vec{w}_i^T(\vec{u} - \vec{b}_0) \end{array}$$

10. Special Case: No Kinematic Excitation



$$c = \sum (\vec{w}_i^T \vec{b}_0 + \chi_i)\, \Lambda_i \geq 0$$

Find $\vec{b}_0$ s.t. $\vec{w}_i^T \vec{b}_0 + \chi_i = 0$

now $\chi_i \equiv 0$

$\Rightarrow \underline{\vec{b}_0 = 0}$

$$\Rightarrow \underline{\underline{W(\vec{b}_0, \Lambda_{i0}) = W(\vec{b}_0, \Lambda_i)}} = \tfrac{1}{2}\|\vec{u}^+\|_M^2 - \tfrac{1}{2}\|\vec{u}^-\|_M^2 = \underline{\underline{T^+ - T^-}}$$

5. Additional kinetic excitation

- Kinetics

$$M(\vec{u}^+ - \vec{u}^-) = \sum \vec{w}_i \Lambda_i + \vec{P} \quad (\Lambda_i \geq 0)$$

$$M(\vec{u}^+ - \vec{u}^- - \vec{\pi}) \in -\mathcal{N}_e \ ; \quad \vec{\pi} := M^{-1}\vec{P}$$

$$\vec{u}^+ \in \vec{z}^- - \mathcal{T}_e^\perp \ ; \quad \vec{z}^- := \vec{u}^- + \vec{\pi}$$

- Kinematics

$$\gamma_i(\vec{u}^+) = \vec{w}_i^T \vec{u}^+ + \chi_i \geq 0 \Leftrightarrow \vec{u}^+ \in \mathcal{P}_e^+$$

- Contact work

$$W = \sum \tfrac{1}{2}(\gamma_i^+ + \gamma_i^-)\Lambda_i$$

$$= \underbrace{\tfrac{1}{2}\|\vec{u}^+ - \vec{b}\|_M^2 - \tfrac{1}{2}\|\vec{z}^- - \vec{b}\|_M^2}_{\equiv -(\vec{u}^+ - \vec{z}^-)^T M \vec{b} + \text{const.}} + \sum \underbrace{\left(\vec{w}_i^T\left(\vec{b} - \tfrac{\vec{\pi}}{2}\right) + \chi_i\right)}_{\gamma_i\left(\vec{b} - \frac{\vec{\pi}}{2}\right) =: S_i(\vec{b})} \Lambda_i$$

- Saddle-point

Minimize $f(\vec{b}) = (\vec{u}^+ - \vec{z}^-)^T M \vec{b} + \mathcal{J}_{\mathcal{Z}_e^+}(\vec{b})$

with $\mathcal{Z}_e^+ = \{\vec{b} \mid S_i(\vec{b}) = \gamma_i(\vec{b} - \tfrac{\vec{\pi}}{2}) \geq 0\} = \mathcal{P}_e^+ + \tfrac{\vec{\pi}}{2}$

- Contact work restriction

$$W(\vec{b}_0(\vec{u}^+), \Lambda_{i0}) \leq \alpha$$

$$\Leftrightarrow \vec{u}^+ \in \bigcap_{\vec{b} \in \mathcal{E}} \mathcal{B}_{\sqrt{2\alpha + \|\vec{z}^- - \vec{b}\|_M^2}}(\vec{b})$$

$\mathcal{E}$: corner points of $\mathcal{Z}_e^+$

- Consistency set ($\alpha = W_p$!!!!)

$$\boxed{\mathcal{S} = \mathcal{P}_e^+ \cap (\vec{z}^- - \mathcal{T}_e^\perp) \bigcap_{\vec{b} \in \mathcal{E}} \mathcal{B}_{\sqrt{2\alpha + \|\vec{z}^- - \vec{b}\|_M^2}}(\vec{b})}$$