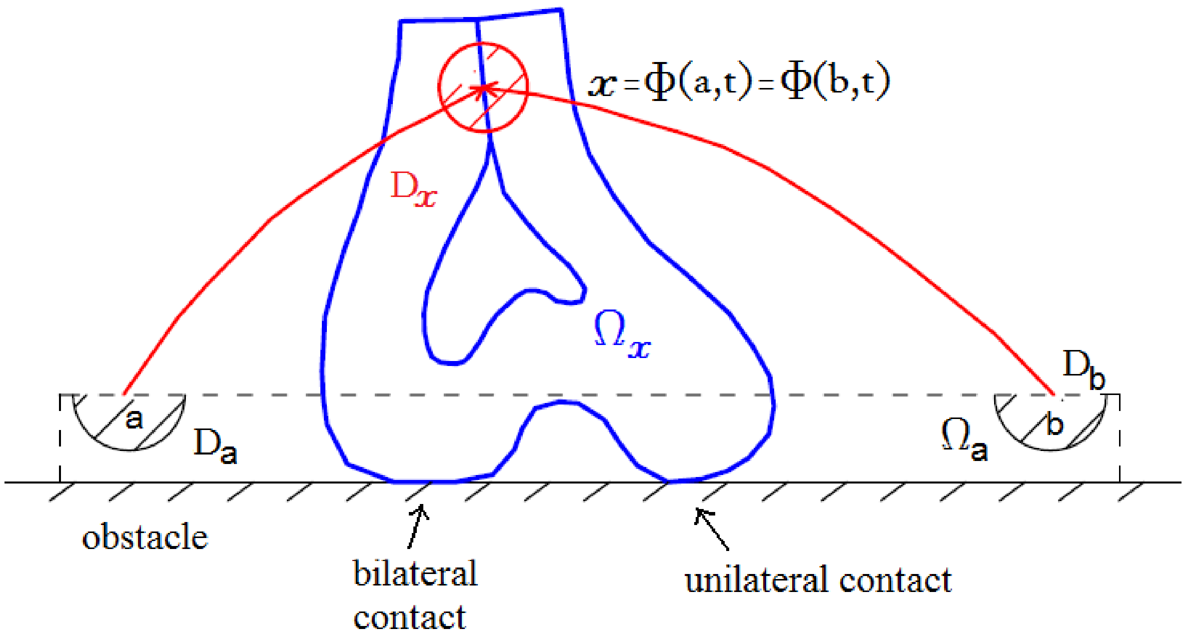
Self-contact, self-collisions and large deformations

Michel Frémond
Francesca Nerilli

Dipartimento di Ingegneria Civile

Università di Roma Tor Vergata



$$x = \Phi(a,t) \qquad x = (x_i) \qquad a = (a_\alpha) = \Phi(a,0)$$

$$\vec{U} = \frac{\partial \Phi}{\partial t}$$ velocity

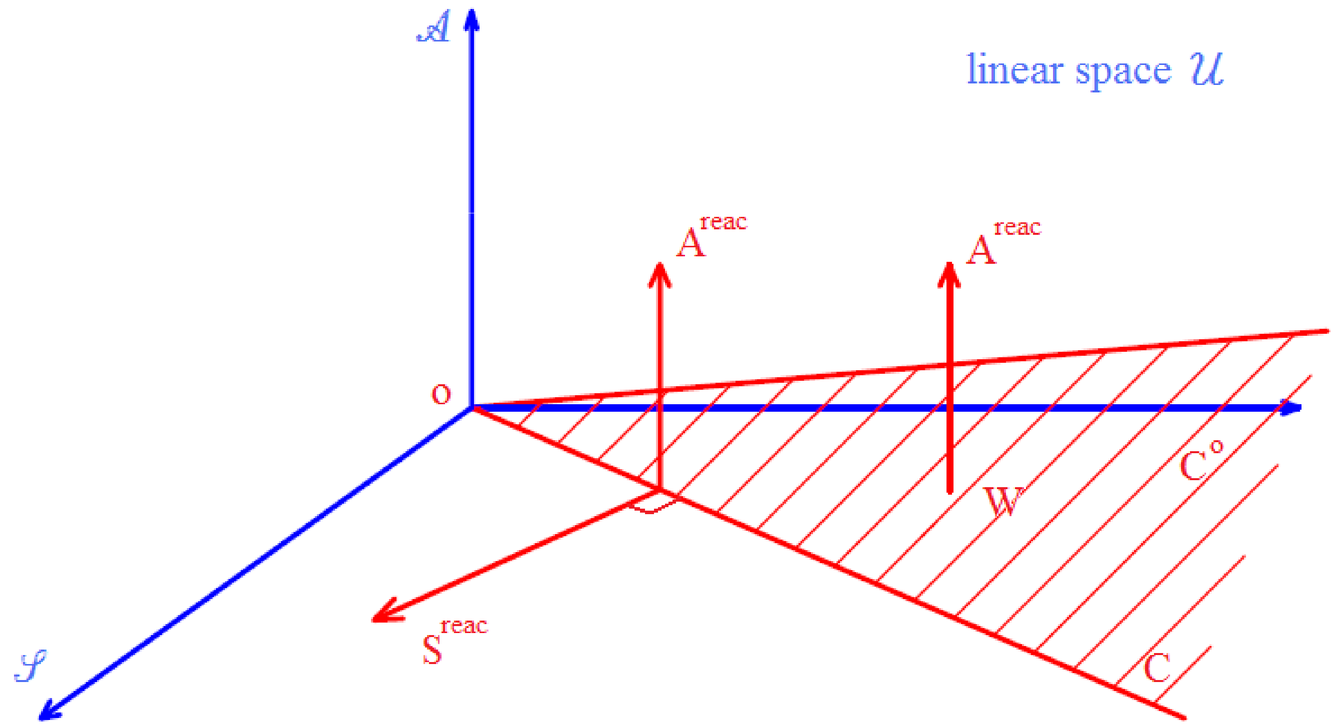
$$F = grad\ \Phi = (F_{i\alpha}) = \frac{\partial \Phi_i}{\partial a_\alpha}$$ gradient matrix

3x3 Matrices

$$A : B = A_{ij} B_{ij}$$

$\begin{cases} \mathscr{S} & \text{symmetric matrices} \\ \mathscr{A} & \text{antisymmetric matrices} \end{cases}$ $\qquad \mathscr{S} \perp \mathscr{A}$

$$C = \{ W \,|\, W \in \mathscr{S} ;\ W \ \textit{is semidefinite positive} \}$$ is a closed convex cone



$$F = RW \ , \quad W \in C \ , \quad RR^{T} = I \ , \quad \det R = 1$$

Classical theory (Elastic behaviour)

$$\Pi = \frac{\partial \psi}{\partial F}, \qquad F = grad \ \Phi$$

Π Boussinesq stress, ψ the free energy, $\psi = \psi(F)$

$$\det F > 0$$

non interpenetration condition.

Rotation matrix R does not intervene; it is impossible that ψ is a convex function of F,

but

there exist mathematical results.

How to take R into account?

VELOCITY OF DEFORMATION

$$grad \ \vec{U}$$

$$\Omega = \frac{\partial R}{\partial t} R^T$$

$$grad \ \Omega = \Omega_\alpha$$

INTERNAL FORCE

$$\Pi$$

$$M \in \mathscr{A}$$

$$\Lambda = (\Lambda_\alpha)$$

EQUATIONS OF MOTION

$$\rho \frac{\partial \vec{U}}{\partial t} = div \ \Pi \qquad in \ \Omega_a$$

$$div \ \Lambda + M = 0 \qquad in \ \Omega_a$$

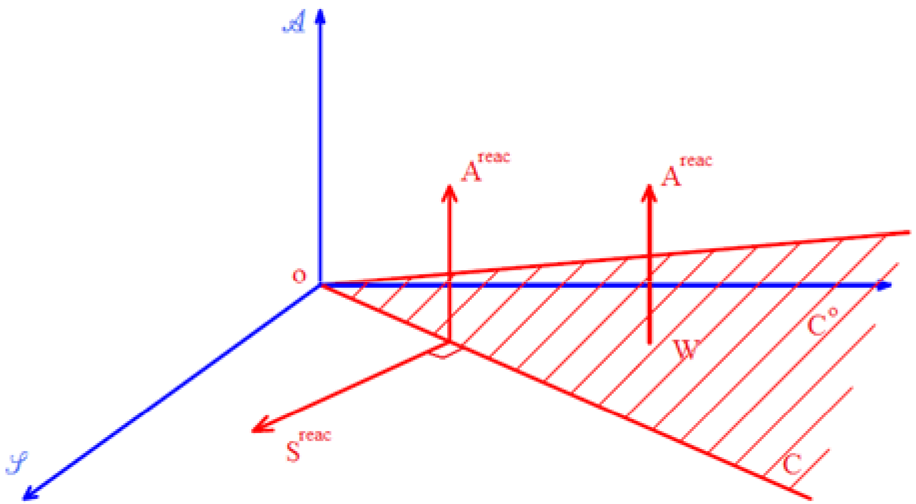
+ B.C. + I.C.

Constitutive laws

W is the physical quantity which describe the elongation.

Impenetrability condition

$$W \in C$$



$$rankW = 3 \longleftrightarrow$$ no flattening

$$rankW = 2 \longleftrightarrow$$ flattening into a surface

$$rankW = 1 \longleftrightarrow$$ flattening into a curve

$$rankW = 0 \longleftrightarrow$$ flattening into a point

Elastic constitutive laws

grad R describes the spacial variation of the rotation matrix.

Free energy

$$\overline{\Psi}(W, gradR) = \Psi(W, \|gradR\|^2) + I_C(W)$$

Constitutive laws

$$\begin{cases} \Pi = R\left(\dfrac{\partial \Psi}{\partial W} + S^{reac} + A^{reac}\right) \\ S^{reac} \in \mathscr{S}, \quad A^{reac} \in \mathscr{A}, \quad S^{reac} + A^{reac} \in \partial I_c(W) \end{cases}$$

$$\Lambda = 4\left(\frac{\partial \Psi}{\partial \|gradR\|^2}\right)(gradR)R^T$$

$$M = \Pi W R^T - R W \Pi^T$$

In this theory

- $(W, gradR) \to \Psi(W, gradR)$ may be convex .

  The effect may be proportional to the cause.

- There exist non unique equilibrium positions in $W^{1,p}(\Omega), \quad p > 3.$

- If $\Psi(W) \to +\infty$ when $\det W \to 0$ there is no flattening. Reaction A^{reac} is present.

- If $\Psi(W) < +\infty$ for $\det W = 0$ flattening is possible. Reaction A^{reac} and S^{reac} are present.

- Constitutive law

$$\Pi = \frac{\partial \Psi}{\partial F} = R \frac{\partial \Psi}{\partial W}$$ is not always valid.

$$\Pi \in R\left(\frac{\partial \Psi}{\partial W} + \partial I_C(W)\right)$$ is always valid.

Self contact

VELOCITY OF DEFORMATION

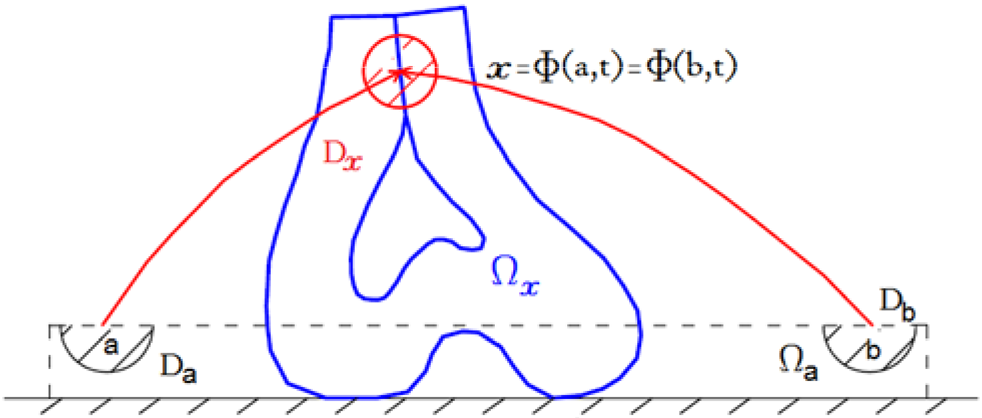
$\vec{U}(a) \quad \vec{U}(b)$

INTERNAL FORCE

$\vec{R}(a) \quad \vec{R}(b)$

EQUATIONS OF MOTION

$$\frac{\vec{R}(a)}{\|cof\ W(a)\|} + \frac{\vec{R}(b)}{\|cof\ W(b)\|} = 0$$



CONSTITUTIVE LAW

$$\vec{R} \in \partial\Phi(\vec{U}(a) - \vec{U}(b))$$

Self collisions

Interior forces become percussions

Constitutive laws involve

$$\frac{\vec{U}^{+}(a)-\vec{U}^{+}(b)+\vec{U}^{-}(a)-\vec{U}^{-}(b)}{2}$$

Collisions when flattening

Boussinesq stress becomes Boussinesq percussion stress

No difficulty: velocity $\vec{U}^{+}$ is uniquely given by velocity $\vec{U}^{-}$

$$rank W = 3$$

$$Unknowns \qquad \Phi(a,t),\ R(a,t),\ A(a,t)$$

$$\rho \frac{\partial^2 \Phi}{\partial t^2} = div \left\{ R\left(\frac{\partial \Psi}{\partial W} + A \right) \right\}$$

$$grad\,\Phi = R\,W(\Phi)$$

$$div\left(\frac{\partial \Psi}{\partial gradR} R^T \right) + R(AW(\Phi) + W(\Phi)A)R^T = 0$$

+ B.C. + I.C.

A is the antisymmetric reaction matrix

$$A \in \partial I_c(W)$$

A schematic example

$$\Psi(W, gradR) = \frac{k}{2}(W - I)^2 + \frac{\hat{k}}{2}\|gradR\|^2 + I_C(W)$$

$$\rho \frac{\partial^2 \Phi}{\partial t^2} = div\{R(k(W(\Phi) - I) + RA\}$$

$$grad\,\Phi = R\,W(\Phi)$$

$$div\left(4\hat{k}(gradR)R^T\right) + R(AW(\Phi) + W(\Phi)A)R^T = 0$$

$$\Pi\vec{N} = \vec{g}, \qquad \Lambda\vec{N} = \vec{m} \qquad on \quad \Gamma_1$$

$$\Phi(a,t) = a, \qquad R = I \qquad on \quad \Gamma_0$$

$$\Phi(a,0) = a, \qquad \frac{\partial \Phi}{\partial t}(a,0) = 0$$

A schematic example

Iterative method: Φ^n, R^n, A^n are known at time $n\Delta t$.

Step 1.

$$\rho \frac{\partial^2 \Phi}{\partial t^2} = div\left\{R^n\left(k\left(W\left(\Phi^n\right) - I\right) + R^n A^n\right\}$$

$\longrightarrow$ give $\boxed{\Phi^{n+1}}$

Step 2.

$$grad\,\Phi^{n+1} = RW\left(\Phi^{n+1}\right)$$

$\longrightarrow$ give $\boxed{R^{n+1}}$

Step 3.

$$div\left(4\hat{k}\left(gradR^{n+1}\right)R^{n+1^T}\right) + R^{n+1}\left(AW\left(\Phi^{n+1}\right) + W\left(\Phi^{n+1}\right)A\right)R^{n+1^T} = 0$$

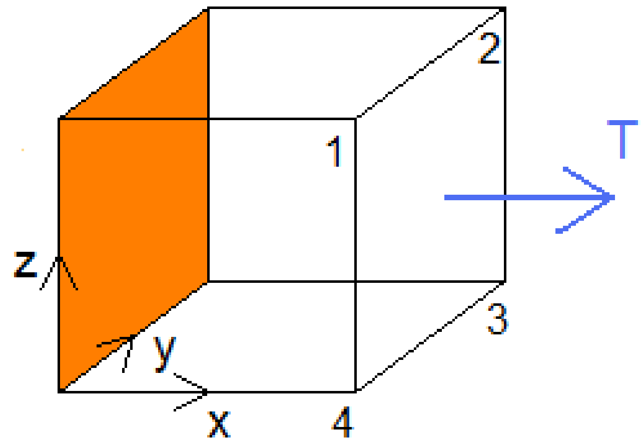
$\longrightarrow$ give $\boxed{A^{n+1}}$

Weak equations and finite elements

Φ, R are piecewise continous

A is either piecewise continous or piecewise constant

Application 1 - Traction



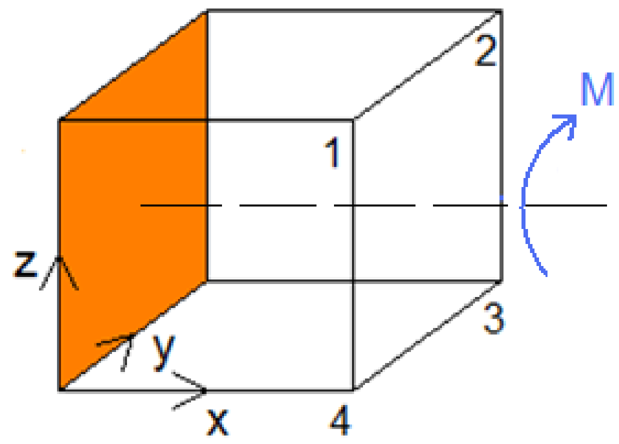
Only elongation $\longrightarrow$ $A = 0$ at each time step

$R = I$ at each time step

k is the elongation rigidity

Periodic motion

Application 2 - Torque



Only torsion $\longrightarrow$ $A \neq 0$ at each time step

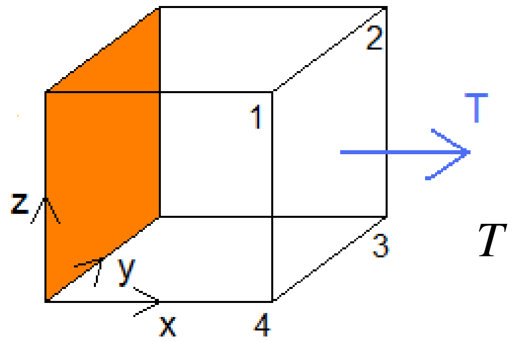
$R \neq I$ at each time step

$\hat{k}$ is the rotation rigidity.

In case $\hat{k} = 0$ (the usual theory) rotation does not have limitation.

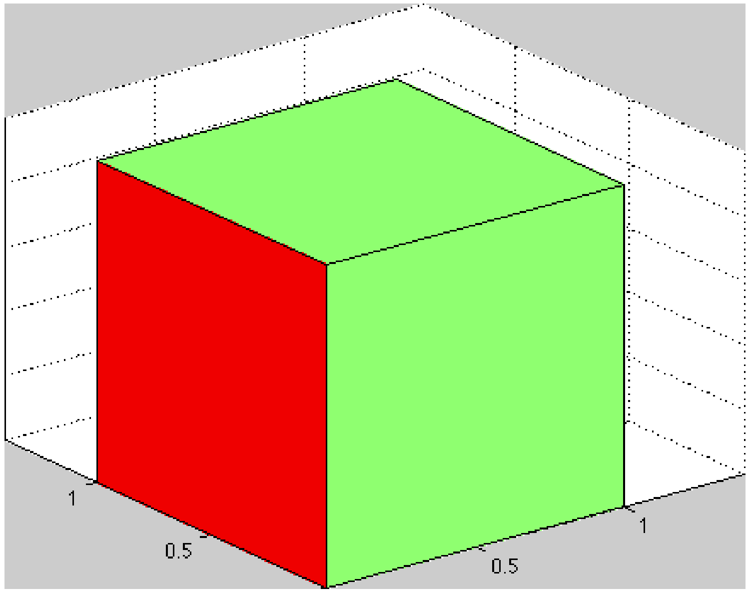
Periodic motion

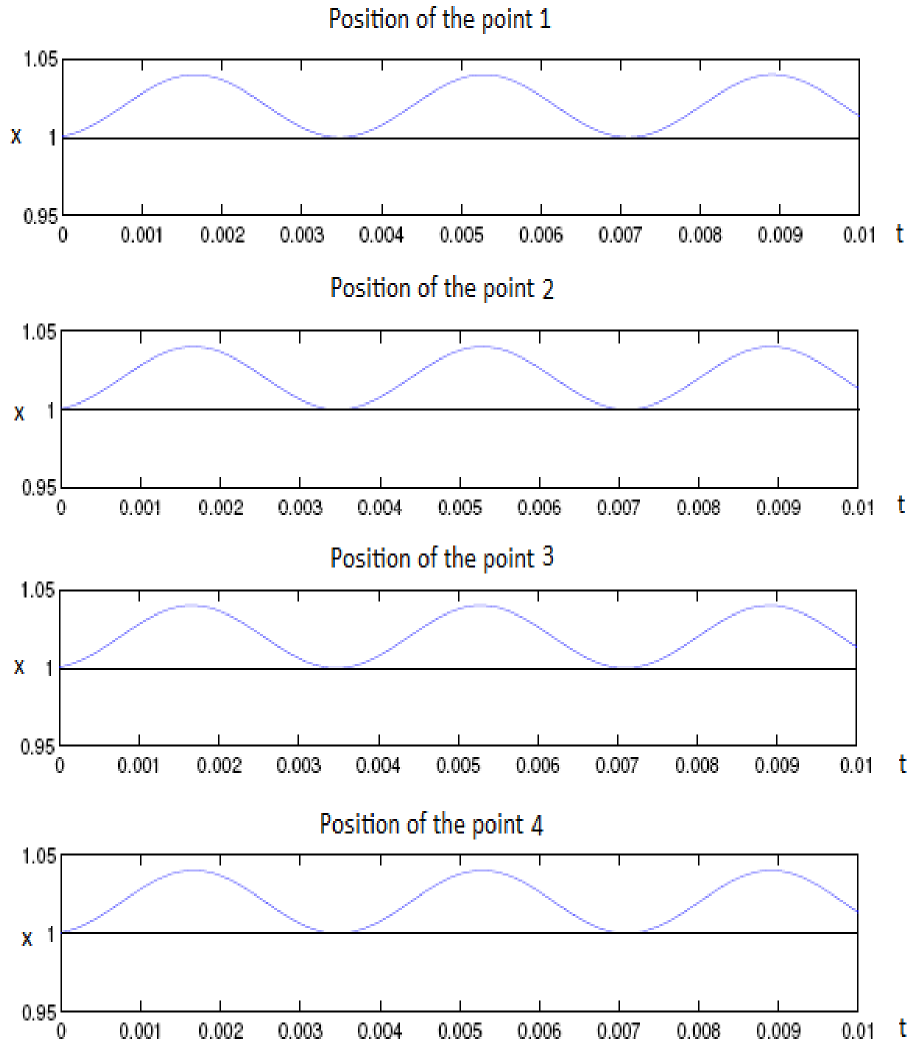
Results



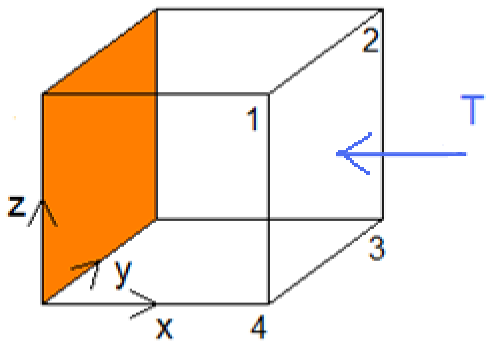
$$T = 2 \cdot 10^{8}\, N$$

$$k = 1 \cdot 10^{10} \frac{N}{m^{2}} \qquad \rho = 1 \cdot 10^{3} \frac{kg}{m^{3}}$$



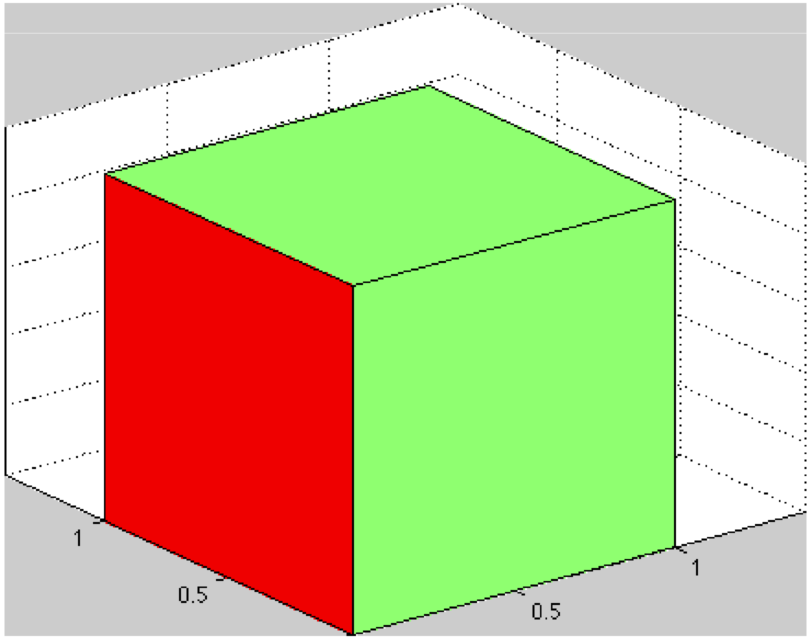


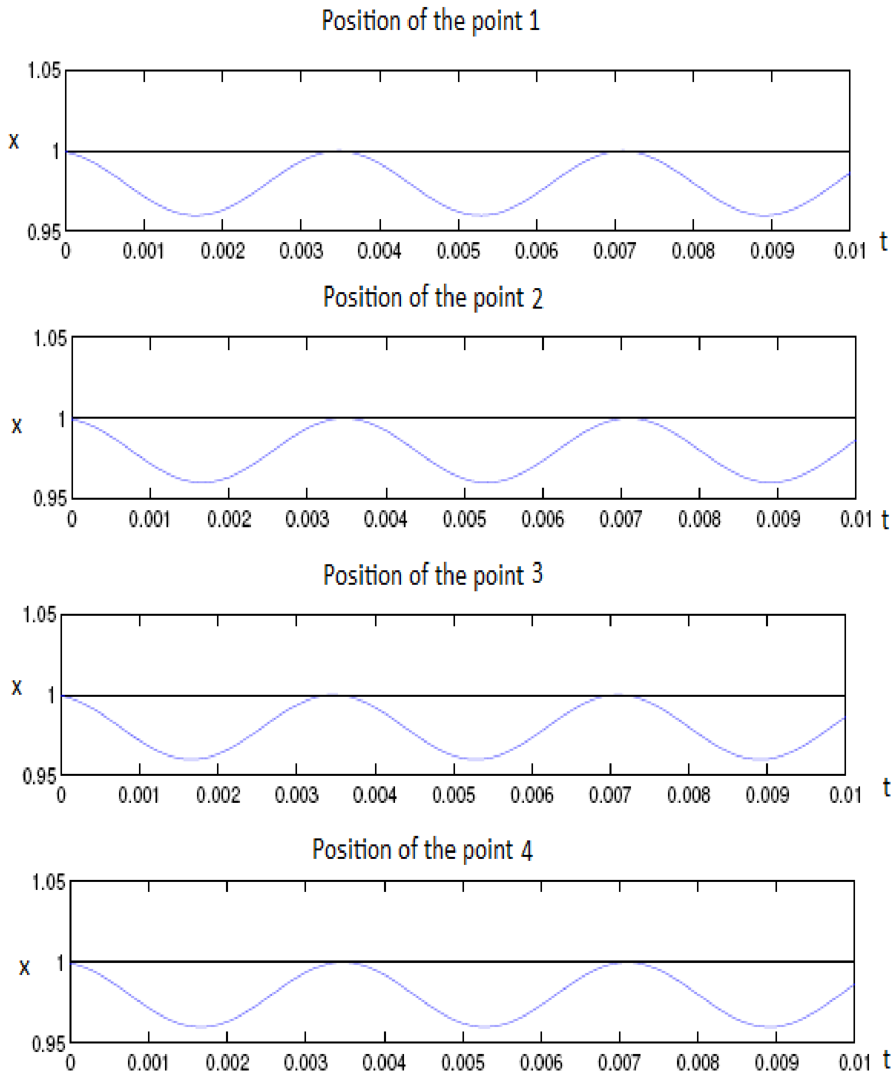
Results



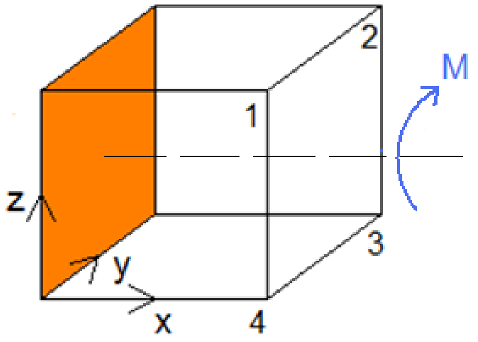
$$T = -2 \cdot 10^{8}\, N$$

$$k = 1 \cdot 10^{10}\, \frac{N}{m^{2}} \qquad \rho = 1 \cdot 10^{3}\, \frac{kg}{m^{3}}$$



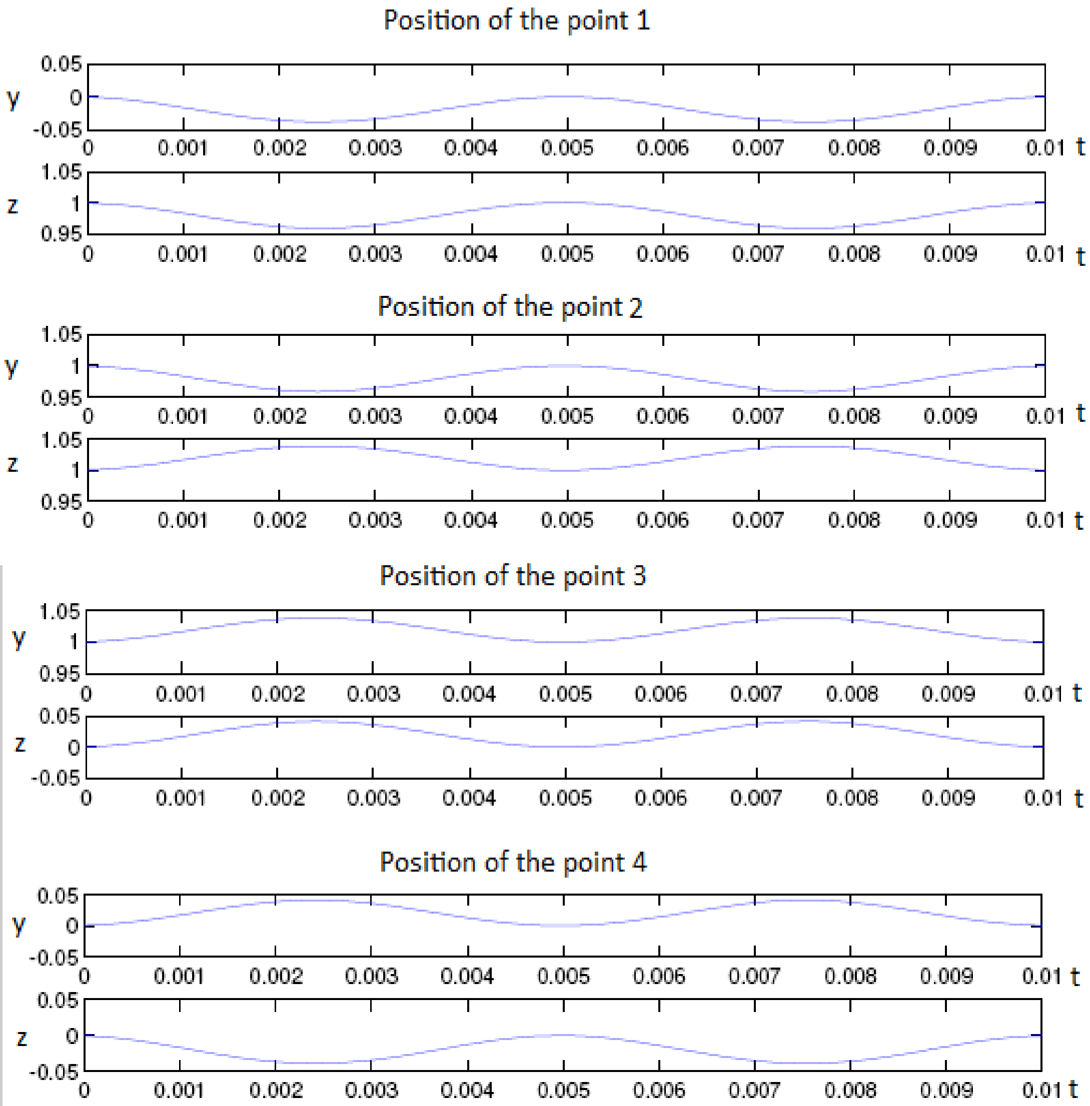


Results

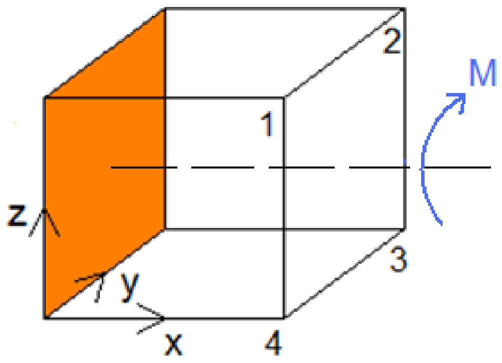


$$\hat{k} = 100$$

$$k = 1 \cdot 10^{10} \frac{N}{m^2} \qquad \rho = 1 \cdot 10^{3} \frac{N}{m^3}$$



Results



$$\hat{k} = 0$$