Analysis of some unilateral conditions with local friction in contact mechanics

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Outline

”Toute action de la nature est produite par elle de la façon la plus directe possible”

Leonardo da Vinci (1452-1519)

1. Introduction

– Unilateral contact conditions - A. Signorini (1933, 1959)
– Existence results for the frictionless case - G. Fichera, J.L. Lions and G. Stampacchia (1970,...)
– Contact problems with nonlocal friction in the static case - G. Duvaut, J.L. Lions (1972,...), J.T. Oden and co-workers (1983,...), M.C. (1984)
– Normal compliance laws - J. Martins and J. T. Oden (1985, 1987)
– Quasistatic problems with normal compliance - L.E. Andersson (1991, 1995, 1999)
– Contact problems with nonlocal friction in the quasistatic case - M.C., E. Pratt and M. Raous (1995, 1996)

– Contact problems with local Coulomb friction in the static case - J. Nečas, J. Jarušek and J. Haslinger (1980), J. Jarušek (1983)
– Numerical analysis of this class of problems - J. Haslinger (1983)
– Quasistatic contact problems with local Coulomb friction law - L.E. Andersson (2000), R. Rocca and M.C. (1999, 2000, 2001)
– Numerical analysis of these problems - R. Rocca and M.C. (2001)
– An interesting "intermediate" static contact problem - P.J. Rabier and O.V. Savin (2000)

Our aim is to extend this model, proposed by P.J. Rabier and O.V. Savin, to the quasistatic case.

2. Quasistatic unilateral contact problems with Coulomb friction

We consider a linear elastic body which occupies the domain Ω of $\mathbb{R}^d$, $d = 2$ or 3, with boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, mes$(\Gamma_1) > 0$, such that the solid is initially in (bounded) contact with local Coulomb friction on Γ_3.

Let

$\boldsymbol{u}$, $u = (u_1, ..., u_d)$, be the displacement field,

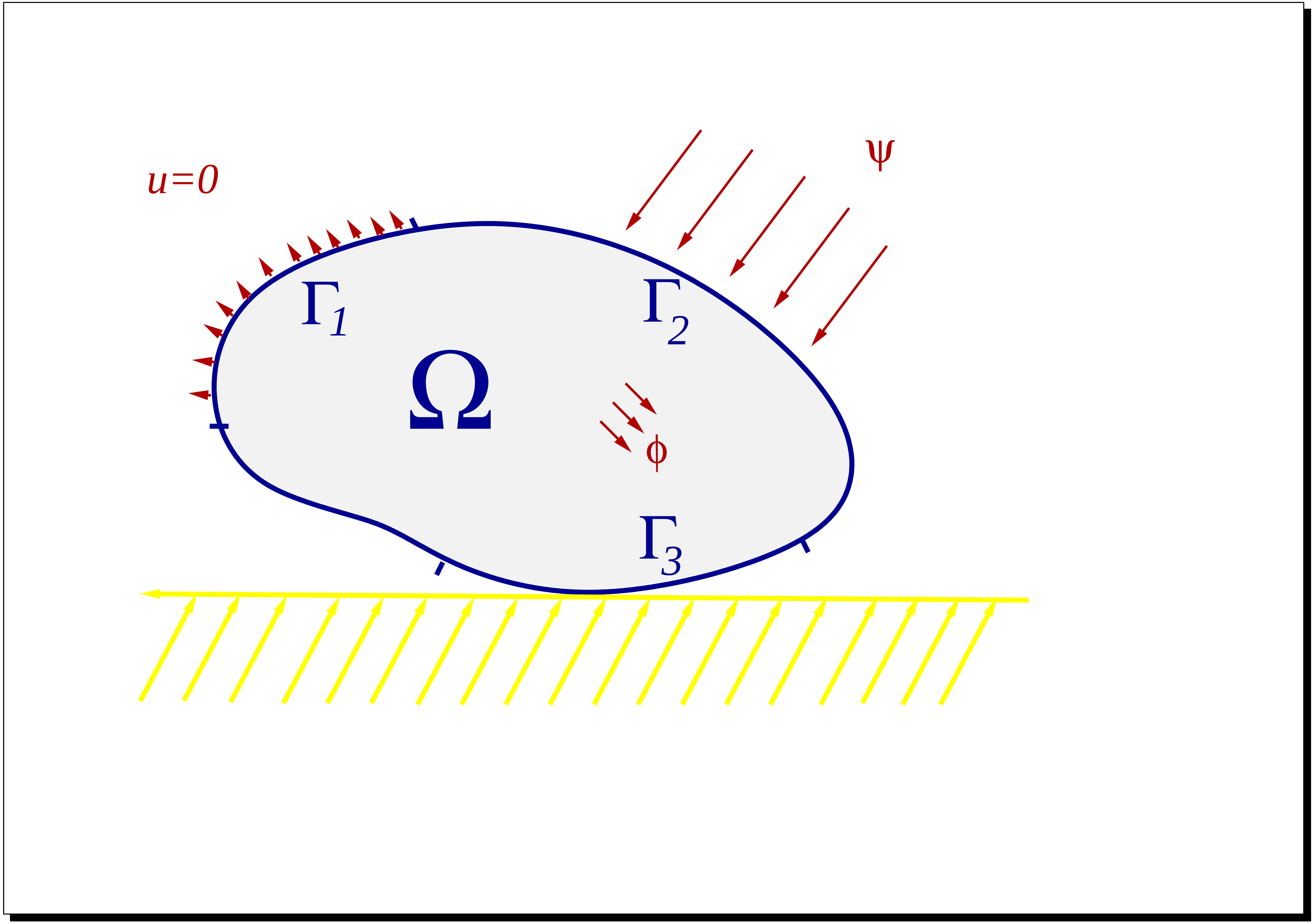
$\boldsymbol{\varepsilon}$, $\varepsilon = (\varepsilon_{kl}(u))$, be the infinitesimal strain tensor,

$\boldsymbol{\mathcal{E}}$, $\mathcal{E} = (a_{klmn})$, be the elasticity tensor verifying the classical conditions of symmetry and ellipticity,

$\boldsymbol{\sigma}$, $\sigma = (\sigma_{kl}(u)) = (a_{klmn}\,\varepsilon_{mn}(u))$, be the stress tensor,

$\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be the given body forces and tractions.

On Γ_1 $\boldsymbol{u} = \boldsymbol{0}$ and in Ω the initial displacements are denoted by $\boldsymbol{u}_0$.



u=0
ψ
Γ_1
Γ_2
Ω
ϕ
Γ_3

We use the decompositions

$$\boldsymbol{u} = u_N \boldsymbol{n} + \boldsymbol{u}_T,\ u_N = \boldsymbol{u} \cdot \boldsymbol{n},$$
$$\boldsymbol{\sigma n} = \sigma_N \boldsymbol{n} + \boldsymbol{\sigma}_T,\ \sigma_N = (\boldsymbol{\sigma n}) \cdot \boldsymbol{n},$$

where $\boldsymbol{n}$, $n = (n_i)$ is the outward unit normal to Γ.

Let us assume that

$\boldsymbol{\phi} \in W^{1,2}(0,T;[L^2(\Omega)]^d),\ \boldsymbol{\psi} \in W^{1,2}(0,T;[L^2(\Gamma)]^d)$,

supp $\boldsymbol{\psi}(t) \subset \overline{\Gamma_2^0} \subset \Gamma_2$ for all $t \in [0,T]$,

$a_{klmn} \in L^\infty(\Omega)$ and $a_{klmn} \in C^{0,1/2+\iota}$ in a neighbourhood of Γ_3,

$0 < \iota \leq 1/2,\ k,l,m,n = 1,...,d$,

$\mu \in L^\infty(\Gamma)$ and is a multiplier on $H^{1/2}(\Gamma)$ of norm $\|\mu\|_{\mathcal{M}}$,

$\mu \geq 0$ a.e. on Γ.

We introduce

$$\boldsymbol{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^d\,;\, \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\},$$
$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V}\,;\, v_N \leq 0 \text{ a.e. on } \Gamma_3\},$$
$$C^{*-} = \{\pi \in H^{-1/2}(\Gamma);\ \ \text{supp}\ \pi \subset \overline{\Gamma}_3,\ \pi \leq 0\},$$
$$(\boldsymbol{L}, \boldsymbol{v}) = (\boldsymbol{\phi}, \boldsymbol{v})_{[L^2(\Omega)]^d} + (\boldsymbol{\psi}, \boldsymbol{v})_{[L^2(\Gamma)]^d} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},$$
$$a(\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega a_{klmn}\varepsilon_{kl}(\boldsymbol{v})\varepsilon_{mn}(\boldsymbol{w})dx \quad \forall\, \boldsymbol{v},\ \boldsymbol{w} \in \boldsymbol{V},$$
$$j_1(\lambda, \boldsymbol{v}) = -\langle \mu\lambda, |\boldsymbol{v}_T|\rangle \quad \forall\, \lambda \in C^{*-}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},$$
$$j_2(\lambda, \boldsymbol{v}) = \int_{\Gamma_3} \mu|\lambda||\boldsymbol{v}_T|ds \quad \forall\, \lambda \in L^2(\Gamma_3), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},$$

where $(\cdot,\cdot)$ is the inner product in $[H^1(\Omega)]^d$, $\langle\ ,\ \rangle$ is the duality product in $[H^{1/2}(\Gamma)]^d$, $[H^{-1/2}(\Gamma)]^d$ and in $H^{1/2}(\Gamma)$, $H^{-1/2}(\Gamma)$.

Let $\boldsymbol{u}_0 \in \boldsymbol{K}$ and $\lambda_0 \in L^\infty(\Gamma_3)$ verify

$\lambda_0 = \sigma_N(\boldsymbol{u}_0) \leq 0$ a.e. on Γ_3 and the compatibility condition

$$a(\boldsymbol{u}_0, \boldsymbol{w} - \boldsymbol{u}_0) + j_2(\lambda_0, \boldsymbol{w}) - j_2(\lambda_0, \boldsymbol{u}_0) \geq (\boldsymbol{L}(0), \boldsymbol{w} - \boldsymbol{u}_0) \quad \forall\, \boldsymbol{w} \in \boldsymbol{K}.$$

On Γ_3 we consider **two types of local friction conditions**.

2.1. Signorini's conditions with local Coulomb friction law

Problem $\mathbf{P_1^c}$: Find $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ satisfying $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in Ω and, for all $t \in]0, T[$, the following equations and conditions:

$$
(P_1^c) \quad \begin{cases}
\text{div } \boldsymbol{\sigma}(\boldsymbol{u}) = -\boldsymbol{\phi} \quad \text{in} \quad \Omega, \qquad \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{\mathcal{E}}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega, \\
\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma_1, \qquad \boldsymbol{\sigma n} = \boldsymbol{\psi} \quad \text{on} \quad \Gamma_2, \\
u_N \leq 0, \qquad \sigma_N \leq 0, \qquad u_N \sigma_N = 0 \quad \text{on} \quad \Gamma_3, \\
|\boldsymbol{\sigma}_T| \leq \mu |\sigma_N| \quad \text{on} \quad \Gamma_3 \\
\quad \text{and} \quad \begin{cases} |\boldsymbol{\sigma}_T| < \mu |\sigma_N| \Rightarrow \dot{\boldsymbol{u}}_T = \mathbf{0}, \\ |\boldsymbol{\sigma}_T| = \mu |\sigma_N| \Rightarrow \exists \lambda \geq 0, \, \dot{\boldsymbol{u}}_T = -\lambda \boldsymbol{\sigma}_T, \end{cases}
\end{cases}
$$

where μ is the coefficient of friction.

We consider the following corresponding variational formulation.

Problem $\mathbf{P_1^v}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ and $\lambda \in W^{1,2}(0,T;H^{-1/2}(\Gamma))$ satisfying $\boldsymbol{u}(0)=\boldsymbol{u}_0$, $\lambda(0)=\lambda_0$, for almost every $t \in]0,T[$ $\lambda(t) \in C^{*-}$ and

$$(P_1^v) \quad \begin{cases} a(\boldsymbol{u},\boldsymbol{v}-\dot{\boldsymbol{u}})+j_1(\lambda,\boldsymbol{v})-j_1(\lambda,\dot{\boldsymbol{u}}) \geq (\boldsymbol{L},\boldsymbol{v}-\dot{\boldsymbol{u}})+\langle \lambda, v_N-\dot{u}_N\rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ \langle \pi-\lambda, u_N\rangle \geq 0 \quad \forall\, \pi \in C^{*-}. \end{cases}$$

The Lagrange multiplier λ verifies the relation $\lambda=\sigma_N(\boldsymbol{u})$.

The results given by L.E. Andersson (2000), R. Rocca et M.C. (1999, 2000, 2001) (including the case coupling adhesion and friction) and the approximation results given by R. Rocca et M.C. (2001) show that under the above assumptions, for sufficiently small values of $\|\mu\|_{L^\infty(\Gamma)}$ and $\|\mu\|_{\mathcal{M}}$, there exists a solution of Problem P_1^v which can be calculated by using a mixed finite element method.

2.2. Modified unilateral conditions with pointwise Coulomb friction law

Problem $\mathbf{P_2^c}$: Find $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ satisfying $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in Ω and, for all $t \in]0, T[$, the following equations and conditions:

$$
(P_2^c) \quad \left\{
\begin{array}{l}
\text{div } \boldsymbol{\sigma}(\boldsymbol{u}) = -\phi \quad \text{in} \quad \Omega, \qquad \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{\mathcal{E}}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega, \\
\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \qquad \boldsymbol{\sigma n} = \boldsymbol{\psi} \quad \text{on} \quad \Gamma_2, \\
\underline{\beta}(u_N) \leq \sigma_N \leq \overline{\beta}(u_N) \quad \text{on} \quad \Gamma_3, \\
|\boldsymbol{\sigma}_T| \leq \mu |\sigma_N| \quad \text{on} \quad \Gamma_3 \\
\quad \text{and} \quad \left\{
\begin{array}{l}
|\boldsymbol{\sigma}_T| < \mu |\sigma_N| \Rightarrow \dot{\boldsymbol{u}}_T = \boldsymbol{0}, \\
|\boldsymbol{\sigma}_T| = \mu |\sigma_N| \Rightarrow \exists \lambda \geq 0, \, \dot{\boldsymbol{u}}_T = -\lambda \boldsymbol{\sigma}_T,
\end{array}
\right.
\end{array}
\right.
$$

where

$\underline{\beta}(r) = 0$ if $r < 0$, $\underline{\beta}(r) = -M$ if $r \geq 0$

and $\overline{\beta}(r) = 0$ if $r \leq 0$, $\overline{\beta}(r) = -M$ if $r > 0$, with $M > 0$ given,

and $\underline{\beta}(u_{0N}) \leq \sigma_N(\boldsymbol{u}_0) \leq \overline{\beta}(u_{0N})$ on Γ_3.

The **Signorini's conditions** correspond to $M = +\infty$.

We assume also the relation $\underline{\beta}(u_{0N}) \le \lambda_0 \le \overline{\beta}(u_{0N})$ a.e. on Γ_3 and we consider the following variational problem:

Problem $\mathbf{P_2^v}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ and $\lambda \in W^{1,2}(0,T;H^{-1/2}(\Gamma))$ satisfying $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\lambda(0) = \lambda_0$, and for almost every $t \in]0,T[$

$$(P_2^v) \quad \begin{cases} a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + j_2(\lambda, \boldsymbol{v}) - j_2(\lambda, \dot{\boldsymbol{u}}) \ge (\boldsymbol{L}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + \langle \lambda, v_N - \dot{u}_N \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ \underline{\beta}(u_N) \le \lambda \le \overline{\beta}(u_N) \quad \text{a.e. on } \Gamma_3. \end{cases}$$

Proposition 2.1. The Problem P_2^v is a variational formulation of Problem P_2^c, where the Lagrange multiplier $\lambda \in L^\infty(\Gamma_3)$ verifies the relation $\lambda = \sigma_N(\boldsymbol{u}) \le 0$ a.e. on Γ_3 and for all $t \in]0,T[$.

3. Incremental formulations, existence and properties of solutions

The Problem P_2^v is solved by using the following incremental formulations.

If $n \in \mathbb{N}^*$, we set $\Delta t := T/n$, $t_i := i\,\Delta t$, $i = 0, 1, ..., n$, if θ is a continuous fonction defined of $t \in [0, T]$ valued in some vector space, we use the notations $\theta^i := \theta(t_i)$ excepting the case $\theta = \boldsymbol{u}$, $\theta = \lambda$, and if η^i, $\forall\, i \in \{0, 1, ..., n\}$, are elements of some vector space, we set

$$\partial\eta^i := \frac{\eta^{i+1} - \eta^i}{\Delta t}, \quad \Delta\eta^i := \eta^{i+1} - \eta^i \quad \forall\, i \in \{0, 1, ..., n-1\}.$$

We then approximate P_2^v using the sequence of following incremental problems $(P_{2,n}^i)_{i=0,1,...,n-1}$.

Problem $\mathbf{P^i_{2,n}}$: Find $\boldsymbol{u}^{i+1} \in \boldsymbol{V}$ and $\lambda^{i+1} \in L^2(\Gamma_3)$ solutions of the system

$$(P_{2,n}^i) \quad \begin{cases} a(\boldsymbol{u}^{i+1}, \boldsymbol{v} - \partial\boldsymbol{u}^i) + j_2(\lambda^{i+1}, \boldsymbol{v}) - j_2(\lambda^{i+1}, \partial\boldsymbol{u}^i) \\ \qquad \geq (\boldsymbol{L}^{i+1}, \boldsymbol{v} - \partial\boldsymbol{u}^i) + \langle \lambda^{i+1}, v_N - \partial u_N^i \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ \underline{\beta}(u_N^{i+1}) \leq \lambda^{i+1} \leq \overline{\beta}(u_N^{i+1}) \ \text{ a.e. on } \Gamma_3. \end{cases}$$

The incremental Problem $P^i_{2,n}$ is equivalent to the following problem, for $i = 0, 1, ..., n-1$.

Problem $\mathbf{Q^i_{2,n}}$: Find $\boldsymbol{u}^{i+1} \in \boldsymbol{V}$ and $\lambda^{i+1} \in L^2(\Gamma_3)$ satisfying

$$(Q^i_{2,n}) \quad \begin{cases} a(\boldsymbol{u}^{i+1}, \boldsymbol{v} - \boldsymbol{u}^{i+1}) + j_2(\lambda^{i+1}, \boldsymbol{v} - \boldsymbol{u}^i) - j_2(\lambda^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ \qquad \geq (\boldsymbol{L}^{i+1}, \boldsymbol{v} - \boldsymbol{u}^{i+1}) + \langle \lambda^{i+1}, v_N - u_N^{i+1} \rangle \quad \forall \, \boldsymbol{v} \in \boldsymbol{V}, \\ \underline{\beta}(u_N^{i+1}) \leq \lambda^{i+1} \leq \overline{\beta}(u_N^{i+1}) \quad \text{a.e. on} \quad \Gamma_3. \end{cases}$$

Theorem 3.1. For every M with $0 < M < +\infty$ there exists a solution $(\boldsymbol{u}^{i+1}, \lambda^{i+1}) \in \boldsymbol{V} \times L^2(\Gamma_3)$ of Problem $Q^i_{2,n}$, $\forall \, i \in \{0, 1, ..., n-1\}$.

Proposition 3.1. There exist $\bar{\mu} > 0$ and $C > 0$, independent of n, such that if $\|\mu\|_{\mathcal{M}} \leq \bar{\mu}$ then the incremental solutions verify the following relations:

$$\|\boldsymbol{u}^{i+1}\|_{[H^1(\Omega)]^d} \leq C \left\{ \|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[H^{-\frac{1}{2}}(\Gamma)]^d} \right\},$$
$$\|\Delta \boldsymbol{u}^{i}\|_{[H^1(\Omega)]^d} \leq C \left\{ \|\Delta \boldsymbol{\phi}^{i}\|_{[L^2(\Omega)]^d} + \|\Delta \boldsymbol{\psi}^{i}\|_{[H^{-\frac{1}{2}}(\Gamma)]^d} \right\} \ \forall\, i \in \{0, 1, ..., n-1\}.$$

These estimates enable us to pass to the limit in $(P^i_{2,n})_{i=0,1,...,n-1}$ as $\Delta t \to 0$ and to obtain the following main result.

Theorem 3.2. If $\|\mu\|_{\mathcal{M}} \leq \bar{\mu}$ then for every M with $0 < M < +\infty$ the Problem P^v_2 admits at least one solution $(\boldsymbol{u}, \lambda) \in W^{1,2}(0,T;\boldsymbol{V}) \times W^{1,2}(0,T;H^{-1/2}(\Gamma))$.

Remarks:

– The solutions $(\lambda^i)_{i=1,2,\dots,n}$ and λ belong to $L^\infty(\Gamma_3)$, which represent a better regularity of the normal component of the stress vector on Γ_3.
– This existence result is established by assuming only that $\|\mu\|_{\mathcal{M}}$ is sufficiently small.

The following interesting result also holds.

Theorem 3.3. Let $(\boldsymbol{u}_M, \lambda_M)$ be a solution of Problem P_2^v corresponding to M. At each time $t \in]0,T[$, we set

$$\Gamma_p(\boldsymbol{u}_M) := \Big\{\boldsymbol{x} \in \Gamma_3;\ u_{M,N}(\boldsymbol{x}) > 0\Big\}.$$

Then for all $\epsilon > 0$ there exists $k > 0$ such that for all M with $0 < M < +\infty$ we have

$$mes(\Gamma_p(\boldsymbol{u}_M)) \leq \epsilon + \frac{k}{M}\|\boldsymbol{L}\|_{[H^1(\Omega)]^d} \quad \forall t \in]0,T[.$$

One can also prove that if there exists a solution $(\boldsymbol{u}, \lambda)$ of the unilateral contact problem with friction P_1^v such that $\lambda \in L^\infty(\Gamma_3)$ then $(\boldsymbol{u}, \lambda)$ is a solution of Problem P_2^v for every M large enough, and if $(\boldsymbol{u}, \lambda)$ is a solution of P_2^v corresponding to 2 distinct values of $M > 0$ then it is also solution of Problem P_1^v.

Hence, the solutions of P_2^v can be seen not only as **approximations** of solutions to P_1^v but also as **generalized solutions** of the unilateral contact problem with local friction (P.J. Rabier et O.V. Savin (2000)).

4. Some generalizations and applications

The previous results can be extended:

– to quasistatic contact problems with friction between two linear (visco)elastic bodies

– to some contact interactions problems, including the problems coupling adhesion and friction, see, for example, L. Cangémi, M. Schryve, M. Raous and M.C.

5. Perspectives

– Nonlinear quasistatic contact problems with friction

– Other complex contact interactions problems

– Hemivariational formulations

– Numerical analysis and solution methods