Non-smooth contact phenomena and surface damage

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The problem

Damage vs. adhesion

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interfaces and discontinuities

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analytical description by use of phase transitions

- in domains: losing rigidity

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- in domains: losing rigidity
- on surfaces: theory of the adhesion between bodies
- fractures: damage energy concentrated on surfaces

  ... how to relate?

The problem

We describe damage

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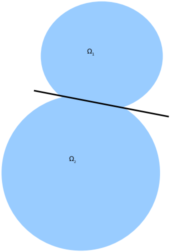
We describe damage
in domains Ω_i which are in contact with adhesion on the surface Γ_c



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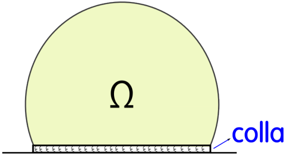

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We describe damage
for a domain Ω which is in contact with adhesion on a rigid support Γ_c (simplifying)

- In the volume domain Ω:
  - small deformations ($\varepsilon(\mathbf{u})$ strain tensor)
  - volume damage (β phase parameter)
  - thermal effects (θ absolute temperature)
- On the contact surface Γ_c:
  - adhesion (χ phase parameter is related to the active bonds of the adhesion $\rightarrow$ "damage parameter")
  - effects of displacement ($\mathbf{u}_{|\Gamma_c}$ trace of the displacement) + effects of volume damage ($\beta_{|\Gamma_c}$ traces of volume damage variables)
  - thermal effects (θ_s absolute temperature)

The damage phenomenon

Macroscopic description of a microscopic phenomenon: changes in the micro-structure $\rightarrow$ degenerating material rigidity

$$\downarrow$$

phase transitions approach

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- $\beta \in [0, 1]$
- $\beta_t \leq 0$ (irreversible phenomenon)

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Internal constraints

To assure physical consistency: free energies and dissipation functionals includes internal constraints

$$\Psi_\Omega = ... + I_{[0,1]}(\beta), \quad \Psi_{\Gamma_c} = ... + I_{[0,1]}(\chi)$$

and (in the case of irreversible damage and/or adhesion evolution)

$$\Phi_\Omega = ... + I_{(-\infty,0]}(\beta_t), \quad \Phi_{\Gamma_c} = ... + I_{(-\infty,0]}(\chi_t)$$

Unilateral conditions

We have to ensure impenetrability between Ω and Γ_c i.e.

$$\mathbf{u}_{|_{\Gamma_c}} \cdot \mathbf{n} \leq 0 \quad \text{in } \Gamma_c.$$

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In the case there is no adhesion $\chi = 0$ the reaction on Γ_c is

$$R = \chi \mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$$

i.e. we have the impenetrability condition as $\mathbf{u} \cdot \mathbf{n} \leq 0$ (cf. also Signorini conditions)

- R is normal to Γ_c
- $R = 0$ if $\mathbf{u} \cdot \mathbf{n} < 0$
- $R = \gamma \mathbf{n}$, $\gamma \geq 0$ if $\mathbf{u} \cdot \mathbf{n} = 0$ (avoiding $\mathbf{u} \cdot \mathbf{n}$ become positive)

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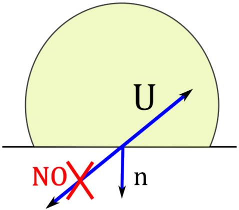

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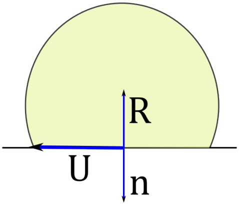

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In the case the adhesion is active $\chi > 0$

$$R = \chi \mathbf{u} + \partial I_{]-\infty,0]}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$$

i.e.

- if $\chi = 0$ we have simply impenetrability conditions
- if $\chi > 0$ there is a reaction avoiding separation

The PDE: the momentum balance

We let

$$\Omega \text{ smooth and } \Gamma := \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_c, t \in (0, T)$$

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We simplify the model taking the displacement u a scalar (hence ∇u stands for deformation)

$$-\text{div}\,(K(\beta)\nabla u) = f \quad \text{in } \Omega$$

where: $K(\beta)$ rigidity depending on the damage parameter
f external volume mechanical action.

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where: $K(\beta) = \beta$ degenerates as damage is complete $\beta = 0$.

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we need some characterization of stress-strain relation once the damage is complete $\beta = 0$

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$$-\text{div}\,(\beta\nabla u + |\nabla u|^2\nabla u) = f \quad \text{in } \Omega$$

we add a nonlinear elastic term in the stress (residual)

The PDE: the momentum balance

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$$\Omega \text{ smooth and } \Gamma := \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_c, t \in (0, T)$$

the boundary conditions for u

$$-\text{div}\,(\beta\nabla u + |\nabla u|^2\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma_1, \quad \sigma\mathbf{n} = g \quad \text{on } \Gamma_2,$$
$$\sigma\mathbf{n} + \chi u_{|\Gamma_c} + \partial I_{]-\infty,0]}(u_{|\Gamma_c}) \ni 0 \quad \text{on } \Gamma_c,$$

where $\chi u_{|\Gamma_c} + \partial I_{]-\infty,0]}(u_{|\Gamma_c})$ on Γ_c is the reaction

The PDE: the evolution of the damage (micro-motions)

We consider in Ω

$$\beta_t - \Delta\beta + \partial I_{[0,1]}(\beta) \ni w - \frac{1}{2}|\nabla(u)|^2$$

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- $\partial I_{[0,1]}(\beta) = 0$ if $\beta \in (0,1)$ and $\partial I_{[0,1]}(0) =]-\infty, 0], \partial I_{[0,1]}(1) = [0, +\infty[$ (a.e. defined) $\rightarrow$ physical constraint
- $w > 0$ is the cohesion
- $-\frac{1}{2}|\nabla(u)|^2$ forces $\beta \searrow 0$

The PDE: the evolution of the damage (micro-motions)

We consider in Ω

$$\beta_t - \Delta\beta + \partial I_{[0,1]}(\beta) \ni w - \frac{1}{2}|\nabla(u)|^2$$

with $\partial_n \beta = -\nu(\beta_{|\Gamma_c} - \chi)$ on Γ_c.

- $\nu(\beta_{|\Gamma_c} - \chi)$ is formally a flux of damage between the body and the glue: it depends on the energy interface coefficient $\nu > 0$

The PDE: the evolution of the adhesion (micro-motions)

We consider in Γ_c

$$\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni w_c - \frac{1}{2}|u_{|\Gamma_c}|^2 + \nu(\beta_{|\Gamma_c} - \chi)$$

$$\partial_n \chi = 0, \quad \text{on } \partial\Gamma_c$$

where

- $\partial I_{[0,1]}(\chi) = 0$ if $\chi \in (0,1)$ and $\partial I_{[0,1]}(0) =]-\infty, 0], \partial I_{[0,1]}(1) = [0, +\infty[$ (a.e. defined) $\rightarrow$ physical constraint
- $w_c > 0$ is the cohesion
- $-\frac{1}{2}|u_{|\Gamma_c}|^2$ forces $\chi \searrow 0$
- $\nu(\beta_{|\Gamma_c} - \chi)$ is formally a flux of damage between the body and the glue: the damage of the glue increases if $\beta_{|\Gamma_c} - \chi < 0$, i.e. if $\chi > \beta_{|\Gamma_c}$, i.e. the glue is less damaged w.r.t. the domain.

The complete PDE system: difficulties

We write the complete PDE system in a variational setting

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$$\int_\Omega (|\nabla u|^2 \nabla u \cdot \nabla v + \beta \nabla u \cdot \nabla v) + \int_{\Gamma_c} \chi u_{|\Gamma_c} v_{|\Gamma_c}$$

$$+ \int_{\Gamma_c} \partial I_-(u_{|\Gamma_c}) v_{|\Gamma_c} = \int_\Omega f v + \int_{\Gamma_2} g v,$$

$v \in W^{1,4}(\Omega), v = 0$ on Γ_1

$$\int_\Omega (\beta_t \phi + \nabla \beta \cdot \nabla \phi) + \int_\Omega \partial I_{[0,1]}(\beta)\phi + \nu \int_{\Gamma_c} (\beta_{|\Gamma_c} - \chi)\phi_{|\Gamma_c}$$

$$= \int_\Omega (w - \frac{1}{2}|\nabla u|^2)\phi, \quad \phi \in H^1(\Omega)$$

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$\rightarrow$ non-smooth constraints on χ, β, and $u_{|\Gamma_c}$

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$\rightarrow$ quadratic nonlinearities

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$$\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni w_c - \frac{1}{2}|u_{|\Gamma_c}|^2 + \nu(\beta_{|\Gamma_c} - \chi) \quad \text{in } \Gamma_c$$

$\rightarrow$ we need sufficient regularity on β and u to control their traces.

The existence theorem

Theorem. Let $T > 0$. Under suitable assumptions on data, there exists a solution (u, χ, β) solving our problem

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Theorem. Let $T > 0$. Under suitable assumptions on data, there exists a solution (u, χ, β) solving our problem with

$$u \in L^\infty(0,T;W^{1,4}(\Omega)), \quad u = 0 \text{ a.e. on } \Gamma_1$$
$$\chi \in H^1(0,T;L^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap L^2(0,T;H^2(\Gamma_c))$$
$$\beta \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$$

Outline

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- approximating procedure

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Remark.

- we need some coercive contribution in the equation for u
- we solve weak versions of the equations in Ω
- Uniqueness is an open problem

Asymptotic analysis as $\nu \to +\infty$

Asymptotic behaviour of solutions as energy interface coefficient $\nu \to +\infty$

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to investigate properties of the adhesion once the

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Asymptotic analysis as $\nu \to +\infty$

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to investigate properties of the adhesion once the

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$$\Psi_{\Gamma_c} = ... + \frac{\nu}{2}|\beta_{|\Gamma_c} - \chi|^2$$

From which it follows (in Γ_c) boundary condition

$$\partial_n \beta = -\nu(\beta_{|\Gamma_c} - \chi)$$

and the source (in the adhesion)

$$\chi_t - \Delta\chi + \partial I_{[0,1]}(\chi) \ni w_c - \frac{1}{2}|u|^2 + \nu(\beta_{|\Gamma_c} - \chi)$$

The estimates

We need a priori estimates on solutions not depending on ν

The estimates

We need a priori estimates on solutions not depending on $\nu \to$ the energy and the dissipation of the system turn out to be (weakly) bounded

$$\|u\|_{L^\infty(0,T;W^{1,4}(\Omega)} \leq c$$

$$\|\beta\|_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))} \leq c$$

$$\|\chi\|_{H^1(0,T;L^2(\Gamma_c))\cap L^\infty(0,T;H^1(\Gamma_c))\cap L^2(0,T;H^2(\Gamma_c))} \leq c$$

$$\|\beta^{1/2}\nabla u\|_{L^\infty(0,T;L^2(\Omega))} \leq c$$

$$\|\chi^{1/2}u\|_{L^\infty(0,T;L^2(\Gamma_c))} \leq c$$

$$\nu^{1/2}\|\beta-\chi\|_{L^\infty(0,T;L^2(\Gamma_c))} \leq c$$

The convergence

We have (for subsequences) weak convergences

$u_\nu \to u$ weak star in $L^\infty(0,T;W^{1,4}(\Omega))$

$\beta_\nu \to \beta$ weak star in $H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$

$\chi_\nu \to \chi$ weak star in

$H^1(0,T;L^2(\Gamma_c)) \cap L^\infty(0,T;H^1(\Gamma_c)) \cap L^2(0,T;H^2(\Gamma_c))$

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and in particular

$$\beta_{|\Gamma_c} = \chi \quad \text{a.e. in } \Gamma_c.$$

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$\chi_\nu \to \chi$ weak star in

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and in particular

$$\beta_{|\Gamma_c} = \chi \quad \text{a.e. in } \Gamma_c.$$

At the limit the body and the adhesion substance have the same physical properties...

The passage to the limit in a weak formulation

The passage to the limit in a weak formulation

For all

$$\phi \in L^2(0,T;H^1(\Omega)), \quad \phi \in [0,1] \text{ a.e.}$$
$$\psi \in L^2(0,T;H^1(\Gamma_c)), \quad \psi \in [0,1] \text{ a.e.}$$

The passage to the limit in a weak formulation

$$\int_0^t \int_\Omega (\beta_t(\beta - \phi) + \nabla\beta \cdot \nabla(\beta - \phi))$$
$$+ \int_0^t \int_\Omega (-w(\beta - \phi) + \frac{1}{2}|\nabla u|^2(\beta - \phi))$$
$$+\nu \int_0^t \int_{\Gamma_c} (\beta_{|\Gamma_c} - \chi)(\beta_{|\Gamma_c} - \phi_{|\Gamma_c}) \leq 0$$

$$\int_0^t \int_{\Gamma_c} (\chi_t(\chi - \psi) + \nabla\chi \cdot \nabla(\chi - \psi))$$
$$+ \int_0^t \int_{\Gamma_c} (-w_c(\chi - \psi) + \frac{1}{2}|u_{|\Gamma_c}|^2(\chi - \psi))$$
$$+\nu \int_0^t \int_{\Gamma_c} (\chi - \beta_{|\Gamma_c})(\chi - \psi) \leq 0$$

The passage to the limit in a weak formulation

We consider the same test function ϕ for the equation in Ω and in Γ_c:

$$\phi, \quad \psi = \phi_{|\Gamma_c}$$

(ϕ sufficiently regular), with $\phi \in [0, 1]$ a.e.

The passage to the limit in a weak formulation

$$\int_0^t \int_\Omega (\beta_t(\beta-\phi) + \nabla\beta\cdot\nabla(\beta-\phi)) \tag{1}$$
$$+ \int_0^t \int_\Omega \left(-w(\beta-\phi) + \frac{1}{2}|\nabla u|^2(\beta-\phi)\right)$$
$$+\nu \int_0^t \int_{\Gamma_c} (\beta_{|\Gamma_c} - \chi)(\beta_{|\Gamma_c} - \phi_{|\Gamma_c}) \leq 0$$

$$\int_0^t \int_{\Gamma_c} (\chi_t(\chi-\phi_{|\Gamma_c}) + \nabla\chi\cdot\nabla(\chi-\phi_{|\Gamma_c}))$$
$$+ \int_0^t \int_{\Gamma_c} \left(-w_c(\chi-\phi_{|\Gamma_c}) + \frac{1}{2}|u_{|\Gamma_c}|^2(\chi-\phi_{|\Gamma_c})\right)$$
$$+\nu \int_0^t \int_{\Gamma_c} (\chi - \beta_{|\Gamma_c})(\chi - \phi_{|\Gamma_c}) \leq 0$$

The passage to the limit in a weak formulation

Adding we get

$$\int_0^t \int_\Omega (\beta_t(\beta - \phi) + \nabla\beta \cdot \nabla(\beta - \phi))$$
$$+ \int_0^t \int_\Omega (-w(\beta - \phi) + \frac{1}{2}|\nabla u|^2(\beta - \phi))$$
$$+ \int_0^t \int_{\Gamma_c} (\chi_t(\chi - \phi_{|\Gamma_c}) + \nabla\chi \cdot \nabla(\chi - \phi_{|\Gamma_c}))$$
$$+ \int_0^t \int_{\Gamma_c} (-w_c(\chi - \phi_{|\Gamma_c}) + \frac{1}{2}|u_{|\Gamma_c}|^2(\chi - \phi_{|\Gamma_c}))$$
$$+ \nu \int_0^t \int_{\Gamma_c} (\beta_{|\Gamma_c} - \chi)^2 \leq 0$$

The passage to the limit in a weak formulation

As $\nu \int_0^t \int_{\Gamma_c} (\beta - \chi)^2 \geq 0$

$$
\begin{aligned}
&\int_0^t \int_\Omega (\beta_t(\beta - \phi) + \nabla\beta \cdot \nabla(\beta - \phi)) \\
&+ \int_0^t \int_\Omega \left(-w(\beta - \phi) + \frac{1}{2}|\nabla u|^2(\beta - \phi)\right) \\
&+ \int_0^t \int_{\Gamma_c} (\chi_t(\chi - \phi_{|\Gamma_c}) + \nabla\chi \cdot \nabla(\chi - \phi_{|\Gamma_c})) \\
&+ \int_0^t \int_{\Gamma_c} \left(-w_c(\chi - \phi_{|\Gamma_c}) + \frac{1}{2}|u_{|\Gamma_c}|^2(\chi - \phi_{|\Gamma_c})\right) \\
&+ \nu \int_0^t \int_{\Gamma_c} (\beta_{|\Gamma_c} - \chi)^2 \leq 0
\end{aligned}
$$

The passage to the limit

Passing to the limit weakly, as $\beta_{|\Gamma_c} = \chi$ a.e. we get

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$$+ \int_0^t \int_\Omega (-w(\beta - \phi) + \frac{1}{2}|\nabla u|^2(\beta - \phi))$$
$$+ \int_0^t \int_{\Gamma_c} ((\beta_{|\Gamma_c})_t(\beta_{|\Gamma_c} - \phi_{|\Gamma_c}) + \nabla\beta_{|\Gamma_c} \cdot \nabla(\beta_{|\Gamma_c} - \phi_{|\Gamma_c}))$$
$$+ \int_0^t \int_{\Gamma_c} -w_c(\beta_{|\Gamma_c} - \phi_{|\Gamma_c}) + \frac{1}{2}|u_{|\Gamma_c}|^2(\beta_{|\Gamma_c} - \phi_{|\Gamma_c}) \leq 0$$

for any ϕ sufficiently regular

$$\phi \in L^2(0, T; H^{3/2}(\Omega)), \quad \phi = 0 \text{ a.e.}$$

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We get a
variational inclusion with dynamic boundary conditions

Ideas and Perspectives

- modelling dynamic boundary conditions
- concentrating damage energy on a surface: relation to fractures models?
- considering thermal properties (inside and on the surface, B. Bonfanti, Rossi, in preparation)
- asymptotic behaviour as the adhesion coefficient $\mu \to +\infty$ in
$$\Psi = ... + \frac{\mu}{2}\chi|u|^2$$
- analysis for two bodies $\Omega_1 \cup \Omega_2 \to$ at the limit we have a unique body Ω with a damage energy concentrated on a surface
- **Numerical simulation by Freddi, Frémond**