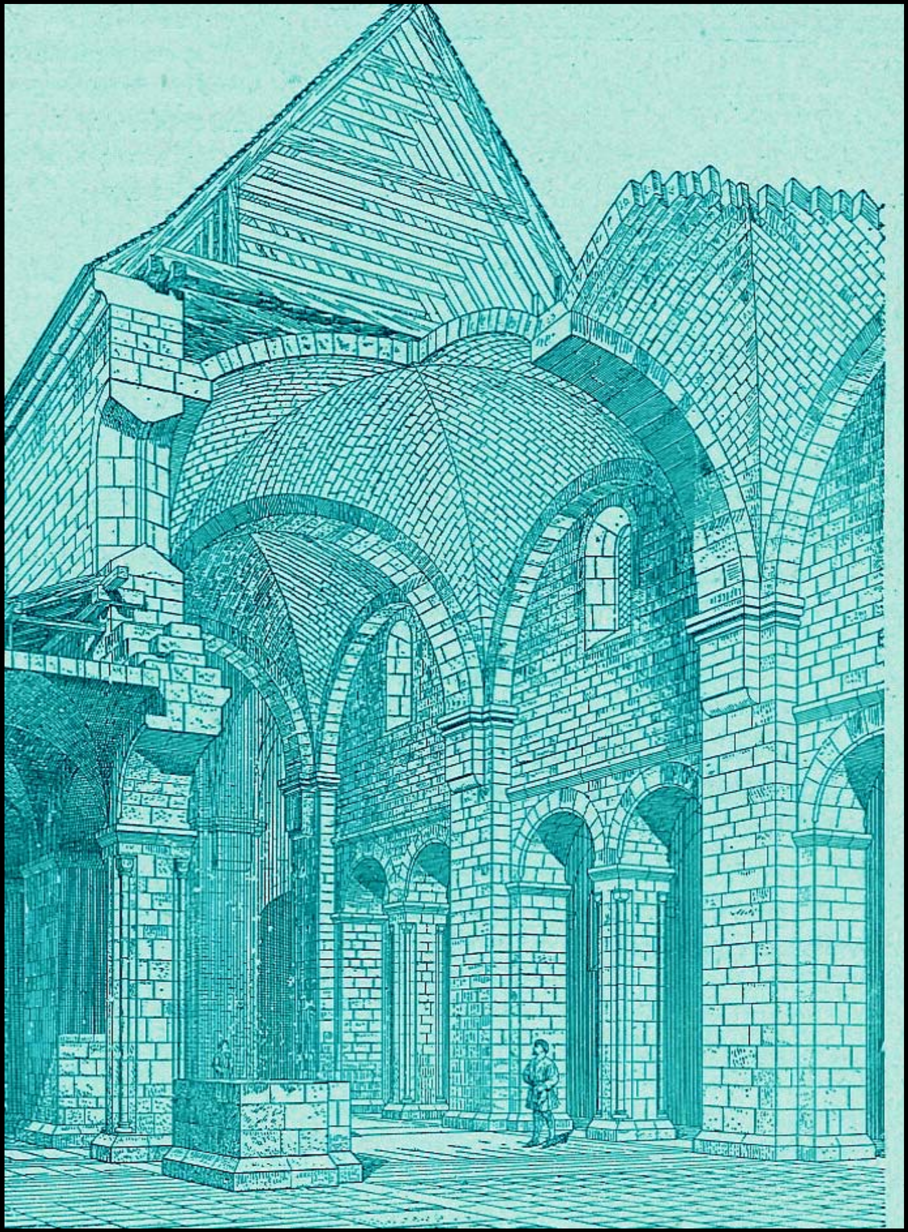
# ON THE STATICS OF NT MASONRY-LIKE VAULTS

by
**A. BARATTA and O. CORBI**
*Department of Structural Engineering,*
*University of Naples, Naples, Italy*

## Abstract

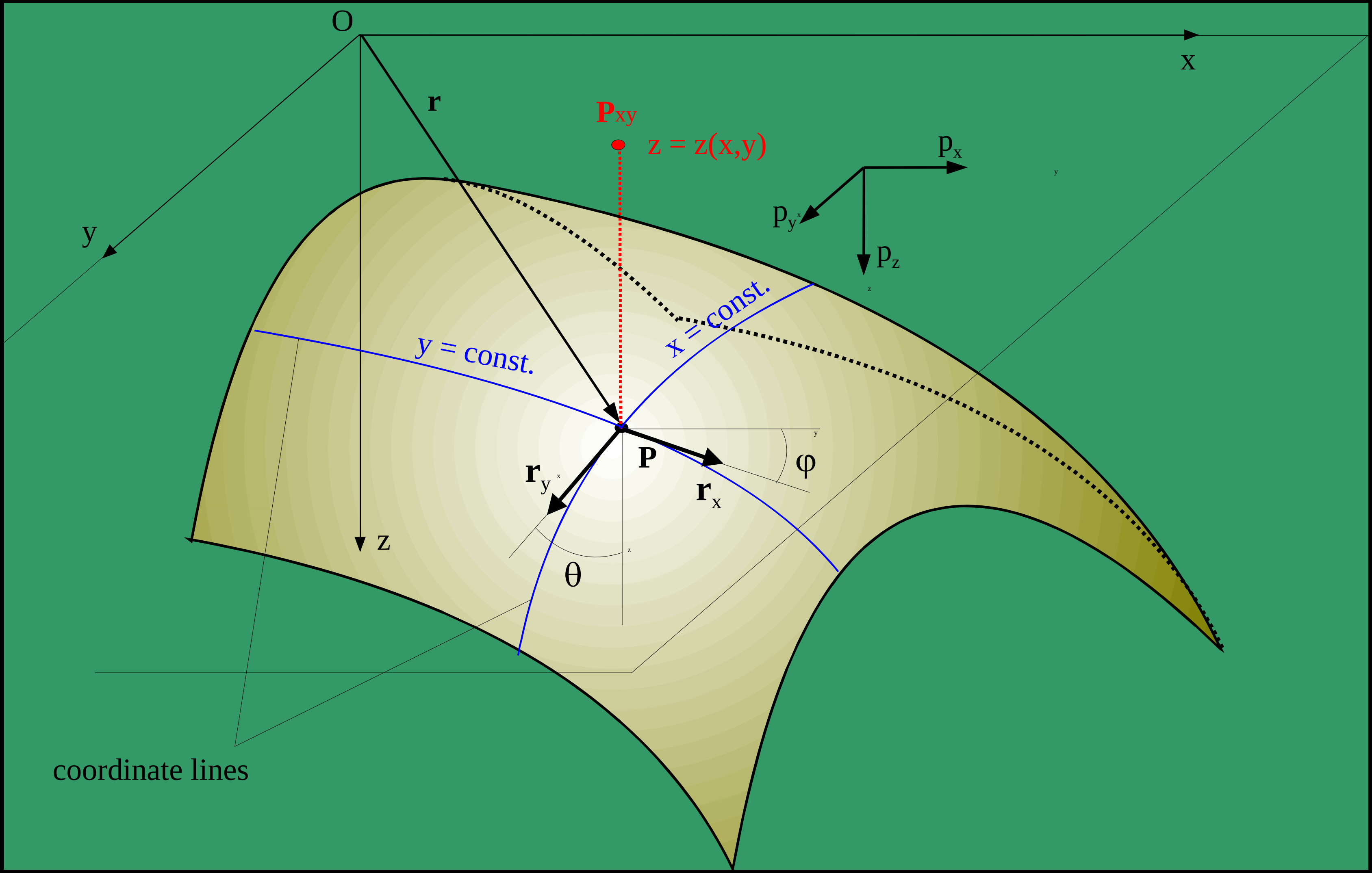
*In the paper a basic approach for dealing with masonry vault analysis is outlined. Actually the fundamental issue in treating such structural typology is, on one side, to account for suitable identification of the vault geometry, which is, as well known, a pretty complex feature of the problem under the point of view of the analytical description especially for vaults of general shape, and, on the other side, to allow the selection of membrane stress surfaces able to equilibrate the applied load and to respect the admissibility conditions of the masonry material.*

*The paper focuses on both these two features, outlining a general approach for analytically handling the problem.*



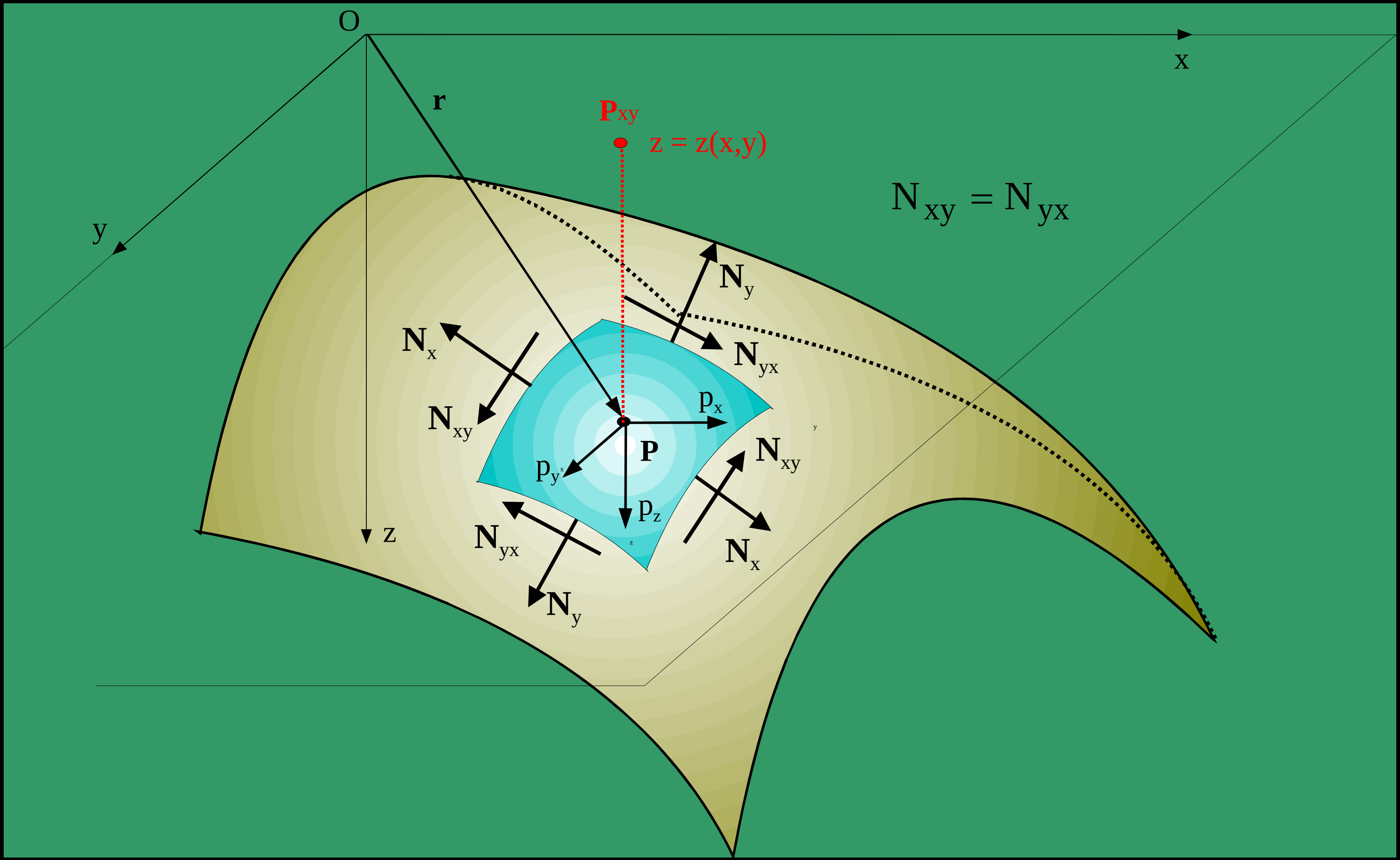
# Statics of shells of general shape

Consider a membrane shell z = z (x,y) subject to generic load components $p_x$, $p_y$, $p_z$



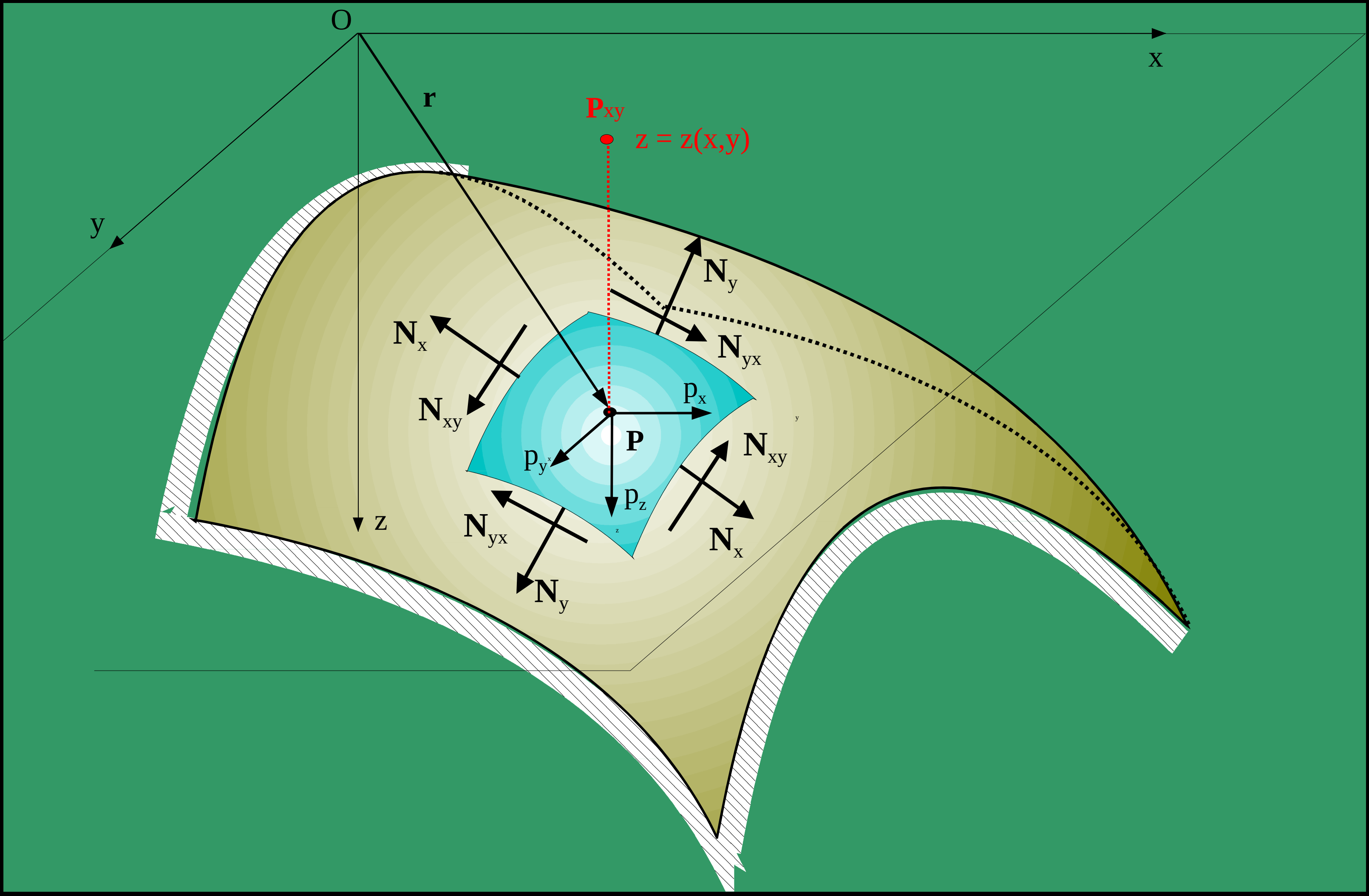
# Statics of shells of general shape

Since the shell is thin, only membrane stresses can be developed



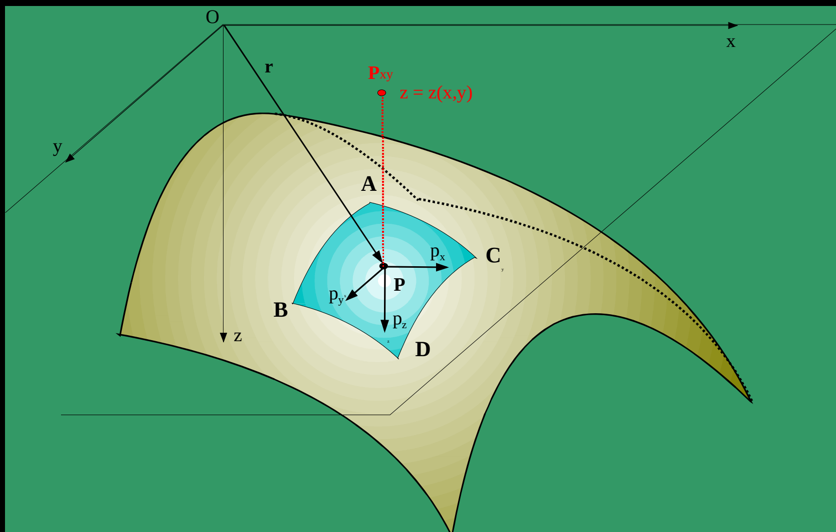
# Statics of shells of general shape

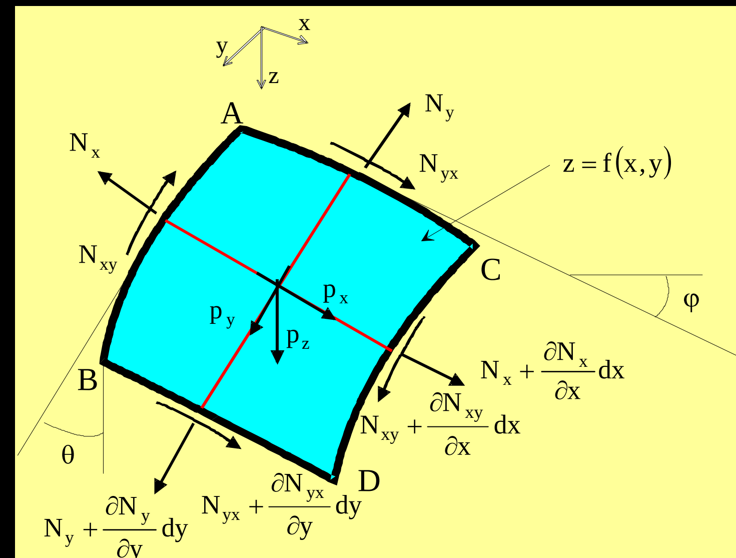
If the shell borders are suitably sustained, any load condition can be carried by membrane stresses



# Statics of shells of general shape

In order to express local equilibrium conditions consider any single surface element, and write equilibrium equations along coordinate axes




For the equilibrium of the element in the x and y directions, one has

$$\left[-N_x\cos\phi+\left(N_x+\frac{\partial N_x}{\partial x}dx\right)\cos\phi''\right]\frac{dy}{\cos\theta'}+\left[-N_{xy}\cos\phi'+\left(N_{xy}+\frac{\partial N_{xy}}{\partial y}dy\right)\cos\phi'\right]\frac{dx}{\cos\phi'}+p_x\frac{dx}{\cos\phi'}\frac{dy}{\cos\theta'}=0$$

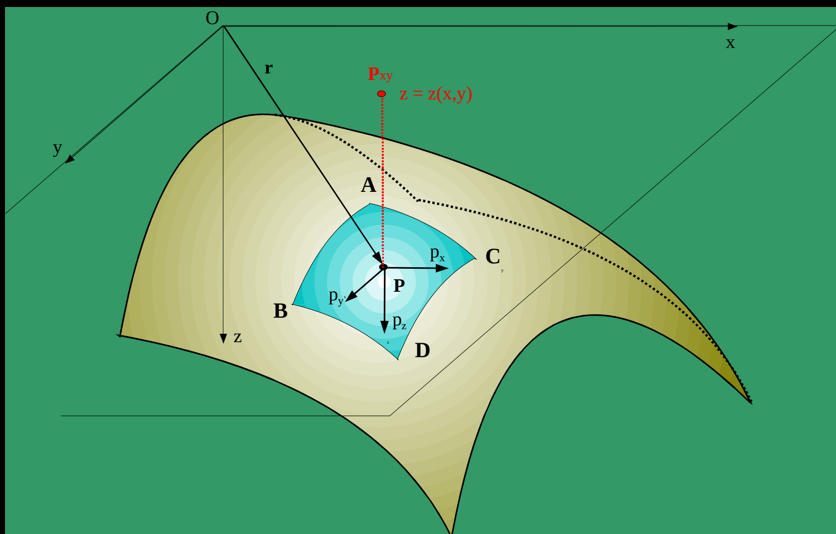
$$\left[-N_{xy}\cos\theta'+\left(N_{xy}+\frac{\partial N_{xy}}{\partial x}dx\right)\cos\theta'\right]\frac{dy}{\cos\theta'}+\left[-N_y\cos\theta+\left(N_y+\frac{\partial N_y}{\partial y}dy\right)\cos\theta''\right]\frac{dx}{\cos\phi'}+p_y\frac{dx}{\cos\phi'}\frac{dy}{\cos\theta'}=0$$

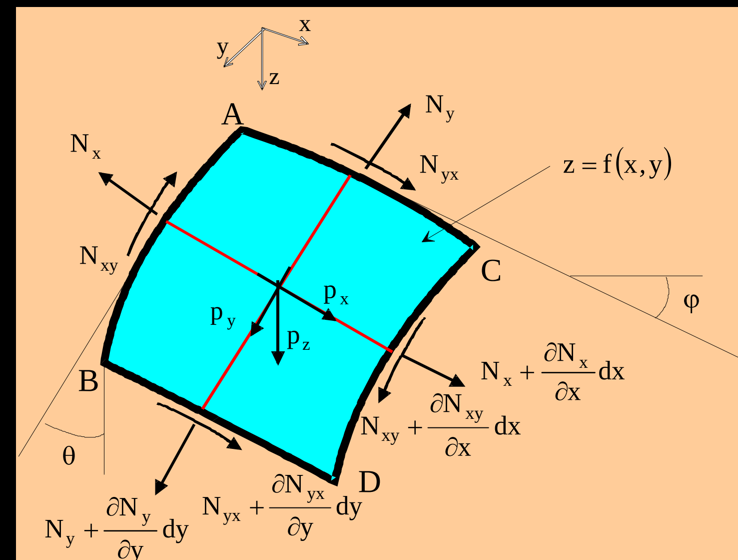
where $\phi$, $\phi'$ and $\phi''$ denote the values assumed by the angle $\phi$, which varies continuously between the two opposite sides AB and CD of the element, respectively on one side AB ($\phi$), at the middle cross-section of AC ($\phi'$) and on the other side CD ($\phi''$) of the single element.

Analogously $\theta$, $\theta'$ and $\theta''$ denote the values assumed by the angle $\theta$, which varies continuously between the two opposite sides AC and BD of the element, respectively on one side AC ($\theta$), at the middle cross-section of AB ($\theta'$) and on the other side BD ($\theta''$) of the single element.

# Statics of shells of general shape

After some algebra, the same equations can be written in the form



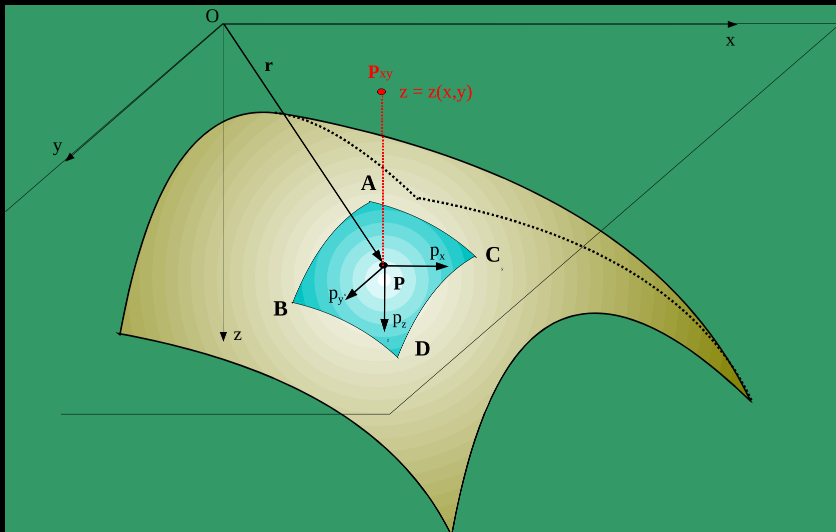



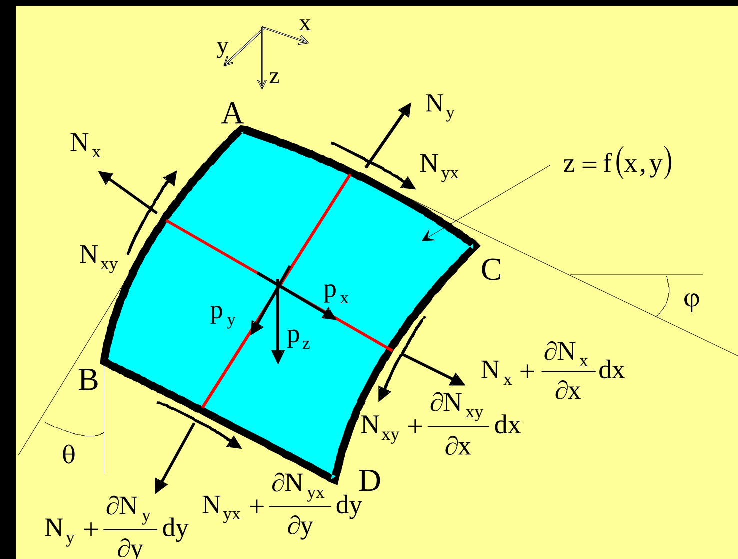

$$\frac{\partial}{\partial x}\left(N_x \frac{\cos\phi}{\cos\theta}\right) + \frac{\partial N_{xy}}{\partial y} = -p_x \frac{1}{\cos\phi}\frac{1}{\cos\theta}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial}{\partial y}\left(N_y \frac{\cos\theta}{\cos\phi}\right) = -p_y \frac{1}{\cos\phi}\frac{1}{\cos\theta}$$

# Statics of shells of general shape

Additionally, for equilibrium along the z-direction one has




$$\left\{\left[-N_x+\left(N_x+\frac{\partial N_x}{\partial x}dx\right)\right]\frac{dy}{\cos\theta}+\left[-N_{xy}+\left(N_{xy}+\frac{\partial N_{xy}}{\partial y}dy\right)\right]\frac{dx}{\cos\phi}\right\}\sin\phi+$$

$$\left\{\left[-N_{xy}+\left(N_{xy}+\frac{\partial N_{xy}}{\partial x}dx\right)\right]\frac{dy}{\cos\theta}+\left[-N_y+\left(N_y+\frac{\partial N_y}{\partial y}dy\right)\right]\frac{dx}{\cos\phi}+\right\}\sin\theta+p_z\frac{dx}{\cos\phi}\frac{dy}{\cos\theta}=0$$

Whence, after some algebra

$$\left(\frac{\partial N_x}{\partial x}\frac{1}{\cos\theta}+\frac{\partial N_{xy}}{\partial y}\frac{1}{\cos\phi}\right)\sin\phi+\left(\frac{\partial N_{xy}}{\partial x}\frac{1}{\cos\theta}+\frac{\partial N_y}{\partial y}\frac{1}{\cos\phi}\right)\sin\theta=-p_z\frac{1}{\cos\phi}\frac{1}{\cos\theta}$$

# Statics of shells of general shape

Consider the projections $\overline{N}_x, \overline{N}_y, \overline{N}_{xy} = \overline{N}_{yx}$ of the membrane stresses onto the xy plane.

The following equations hold

$$N_x = \overline{N}_x \frac{\cos\theta}{\cos\phi}$$

$$N_y = \overline{N}_y \frac{\cos\phi}{\cos\theta}$$

$$N_{xy} = \overline{N}_{xy}$$

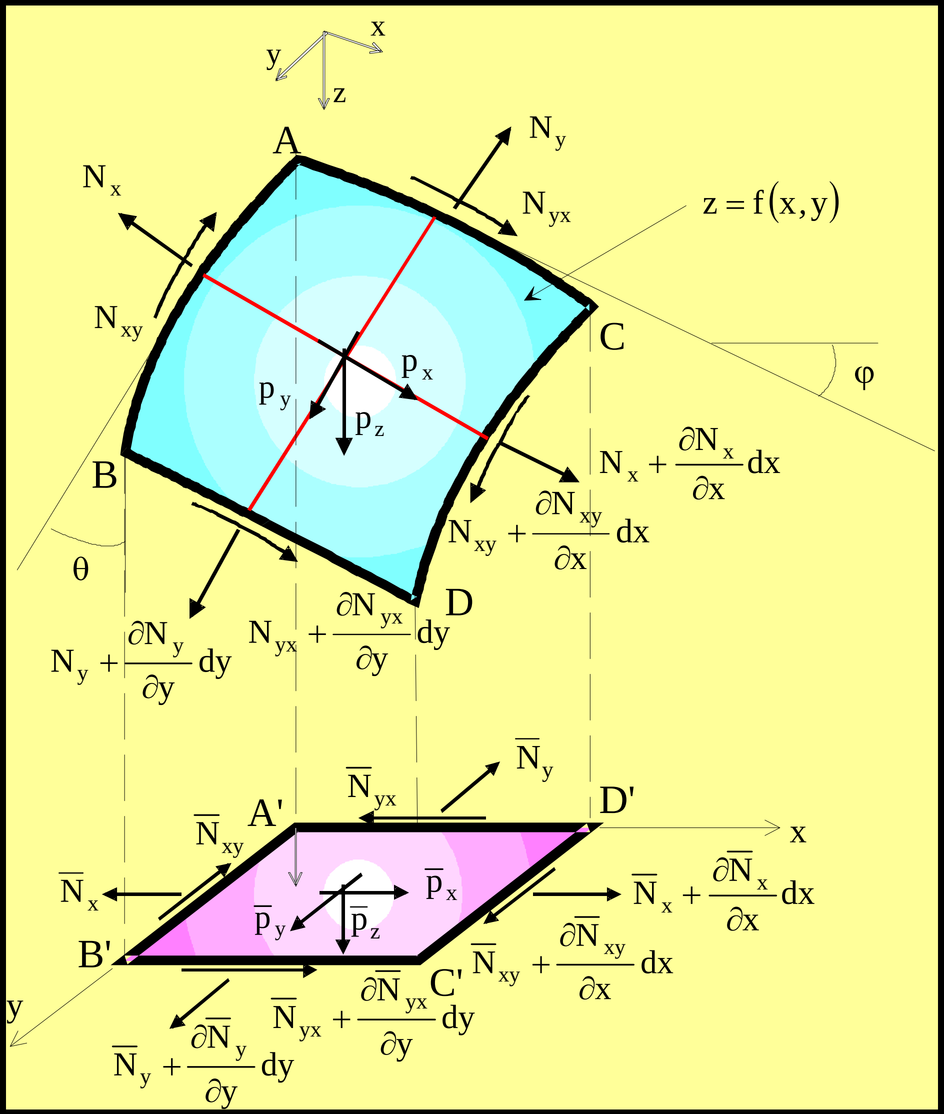
with

$$\tan\phi = \frac{\partial z}{\partial x}, \quad \tan\theta = \frac{\partial z}{\partial y}.$$

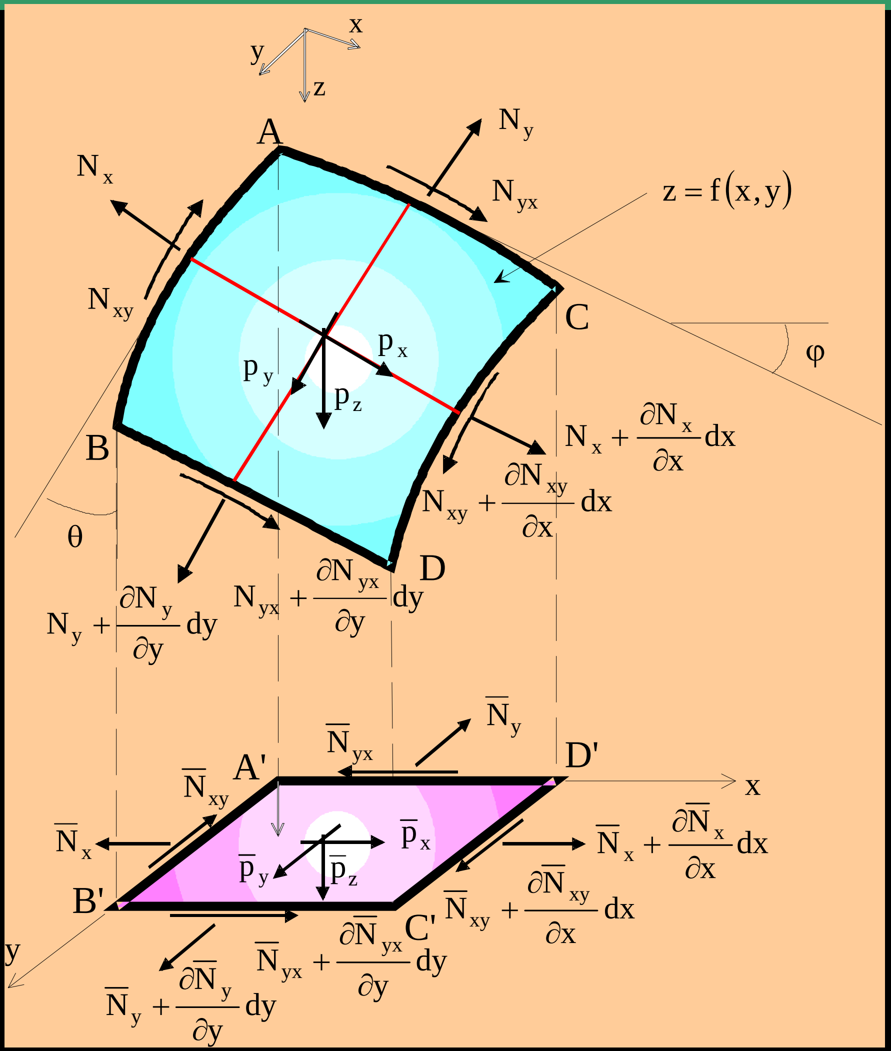
Put moreover

$$p_x = \overline{p}_x \cos\theta \cos\phi$$

$$p_y = \overline{p}_y \cos\theta \cos\phi$$



# Statics of shells of general shape




The equilibrium equations can be written in function of the new variables

$$\frac{\partial \bar{N}_x}{\partial x} + \frac{\partial \bar{N}_{xy}}{\partial y} = -\bar{p}_x$$

$$\frac{\partial \bar{N}_{xy}}{\partial x} + \frac{\partial \bar{N}_y}{\partial y} = -\bar{p}_y$$

The z-equation after some algebra reduces to

$$\bar{N}_x \frac{\partial^2 z}{\partial x^2} + 2\bar{N}_{xy} \frac{\partial^2 z}{\partial x\, \partial y} + \bar{N}_y \frac{\partial^2 z}{\partial y^2} = -\bar{p}_z + \bar{p}_x \frac{\partial z}{\partial x} + \bar{p}_y \frac{\partial z}{\partial y}$$

So the membrane equilibrium becomes analogous to the problem of a linearly elastic plane panel

# Statics of shells of general shape

In most cases, it is advantageous to introduce a stress function $\Psi(x, y)$ that reduces equilibrium equations to one second-order equation as follows.

By analogy with plate problems the equilibrium conditions in the x and y directions are identically verified if one puts

$$\overline{N}_x = \frac{\partial^2 \psi}{\partial y^2} - \int \overline{p}_x dx, \quad \overline{N}_y = \frac{\partial^2 \psi}{\partial x^2} - \int \overline{p}_y dy, \quad \overline{N}_{xy} = \overline{N}_{yx} = -\frac{\partial^2 \psi}{\partial x \, \partial y}.$$

The z-equation after some algebra reduces to

$$\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 \psi}{\partial x \, \partial y}\frac{\partial^2 z}{\partial x \, \partial y} + \frac{\partial^2 \psi}{\partial y^2}\frac{\partial^2 z}{\partial x^2} = -\overline{p}_z + \overline{p}_x \frac{\partial z}{\partial x} + \overline{p}_y \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x^2}\int \overline{p}_x dx + \frac{\partial^2 z}{\partial y^2}\int \overline{p}_y dy$$

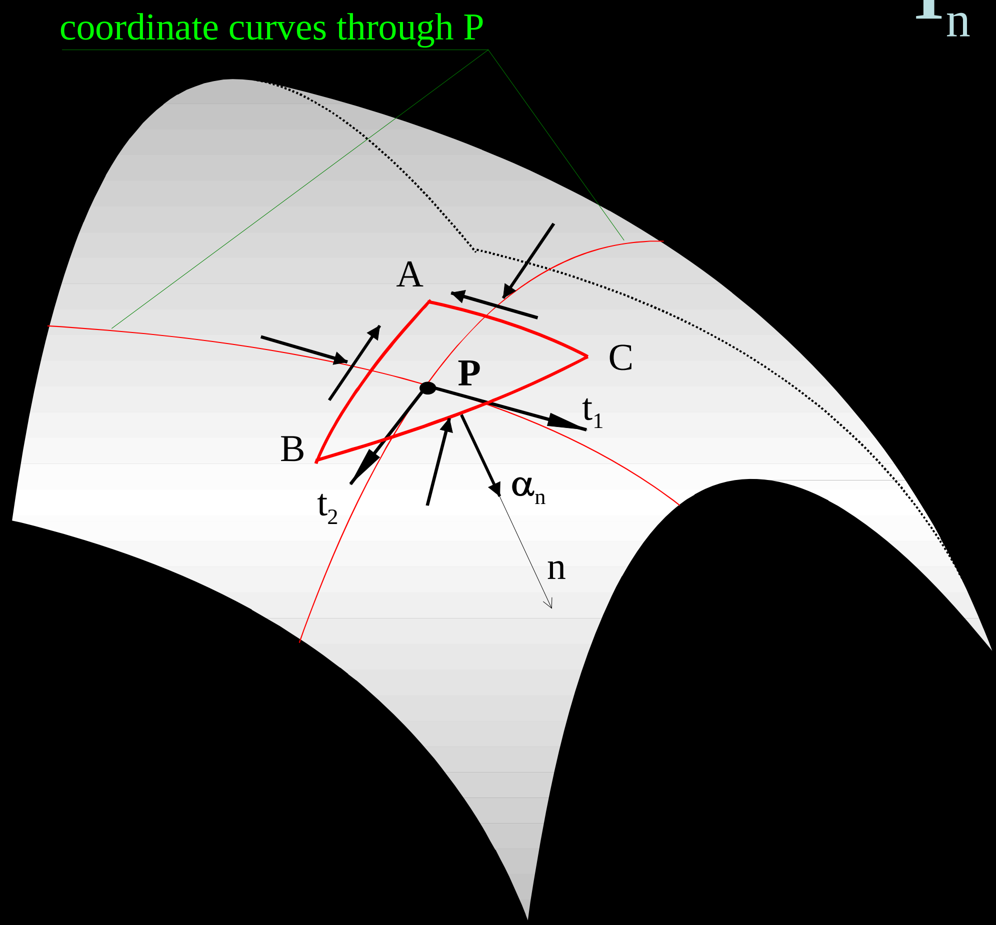
The solution of the problem is thus reduced to the determination of stress function.

If $p_x = p_y = 0$, the latter equation simplifies to:

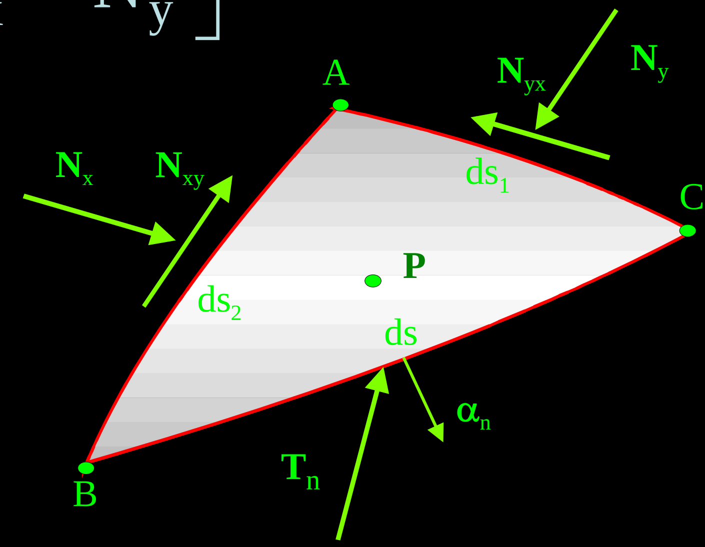
$$\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 \psi}{\partial x \, \partial y}\frac{\partial^2 z}{\partial x \, \partial y} + \frac{\partial^2 \psi}{\partial y^2}\frac{\partial^2 z}{\partial x^2} = -\overline{p}_z$$

# Statics of shells of general shape

## No-tension membrane




$$\mathbf{T}_n = \begin{bmatrix} N_x & N_{xy} \\ N_{yx} & N_y \end{bmatrix} \alpha_n$$



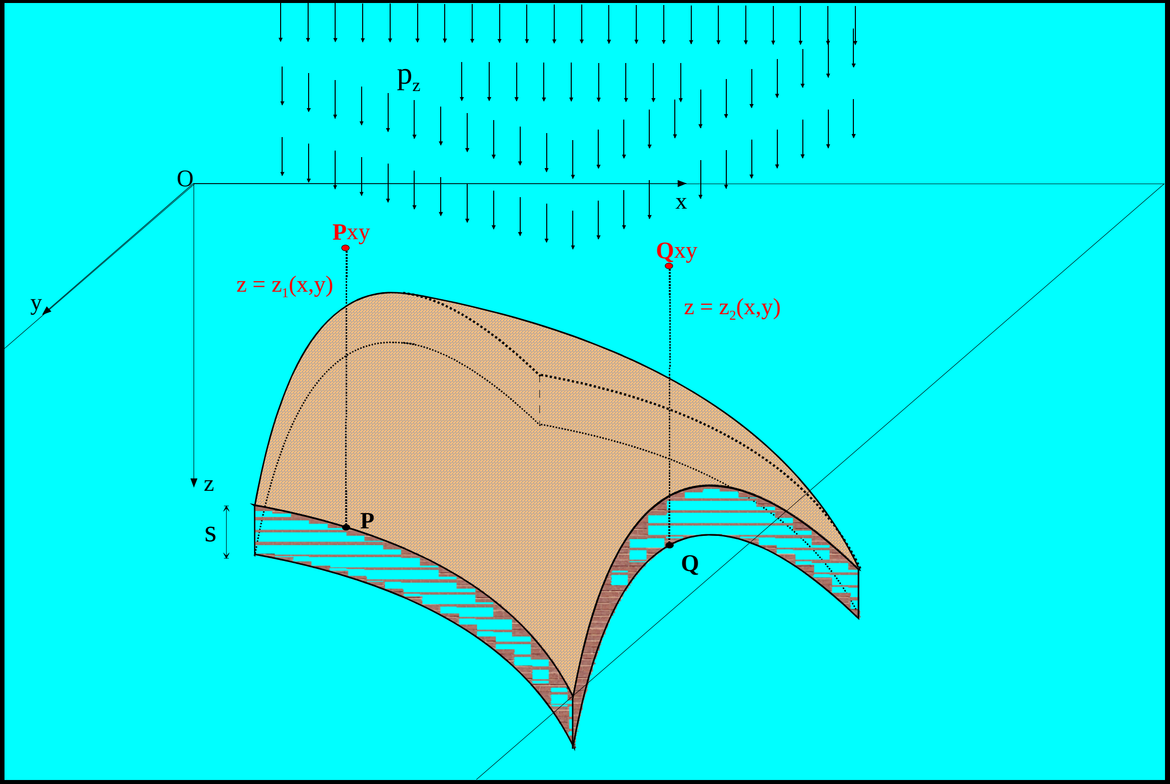

$$N_x \le 0$$

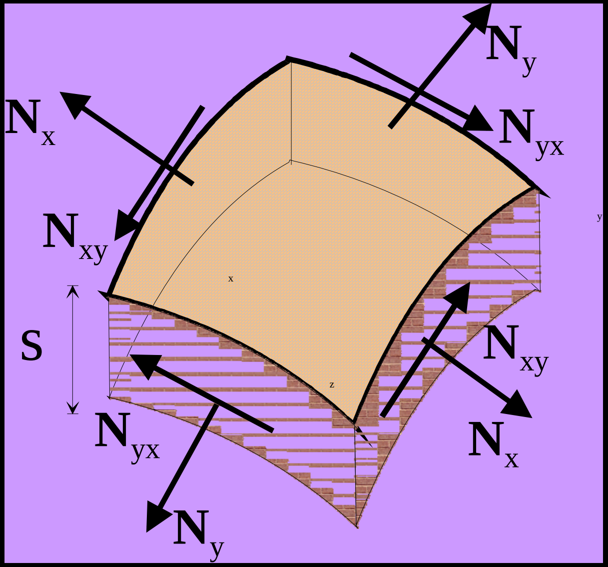
$$N_y \le 0$$

$$N_{xy}^2 - N_x N_y \le 0$$

# Masonry vaults of general shape

Consider a masonry vault with thickness "s", subject only to vertical load.




Since it is assumed that masonry cannot resist tensile stresses, the internal forces have to verify the following admissibility conditions

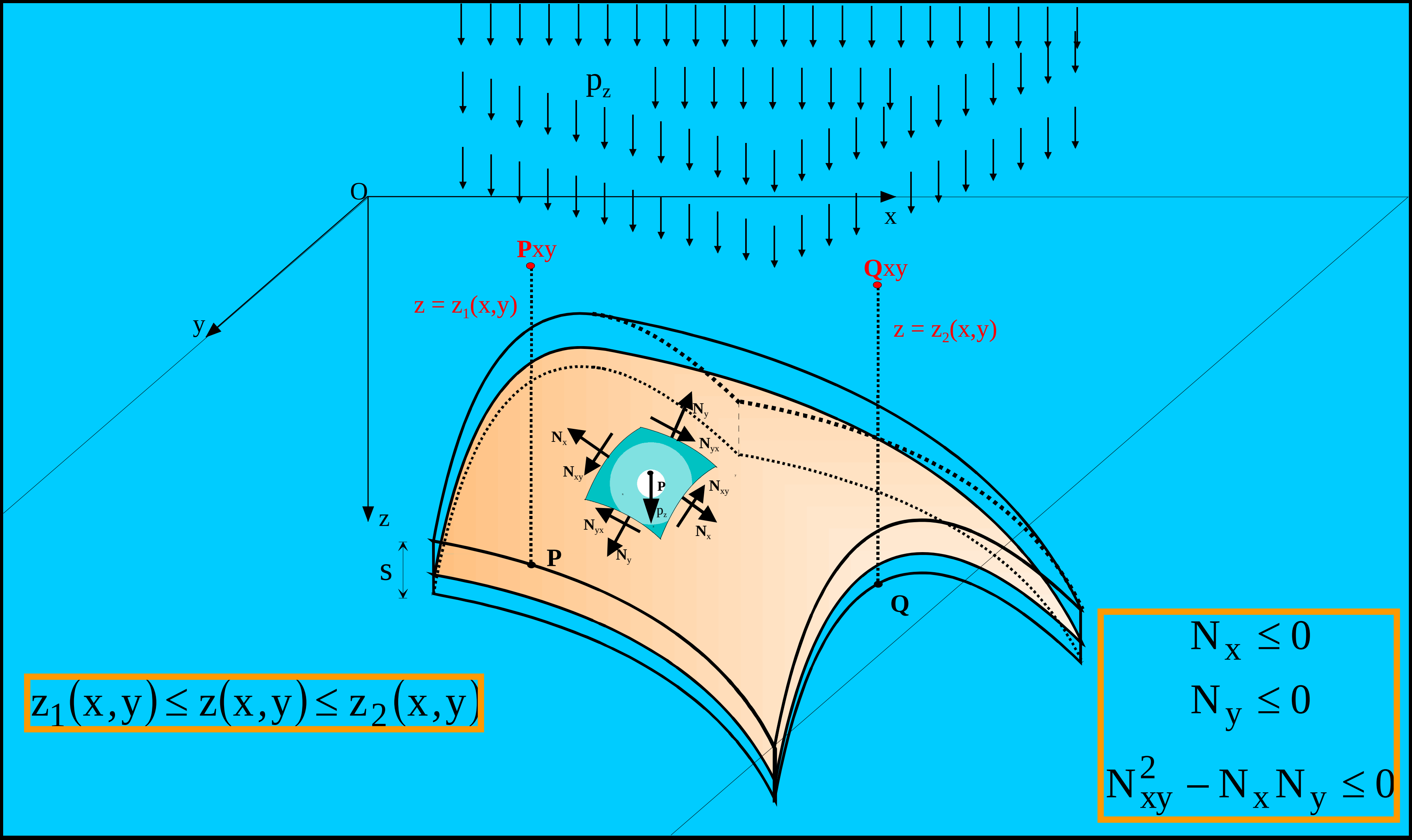
$$N_x \leq 0$$

$$N_y \leq 0$$

$$N_{xy}^2 - N_x N_y \leq 0$$

# Masonry vaults of general shape

A solution can be attempted searching for a membrane surface $z = z(x,y)$ completely internal to the mass of the vault, resisting the downward (i.e. positive) load $p_z$ by purely compressive internal forces

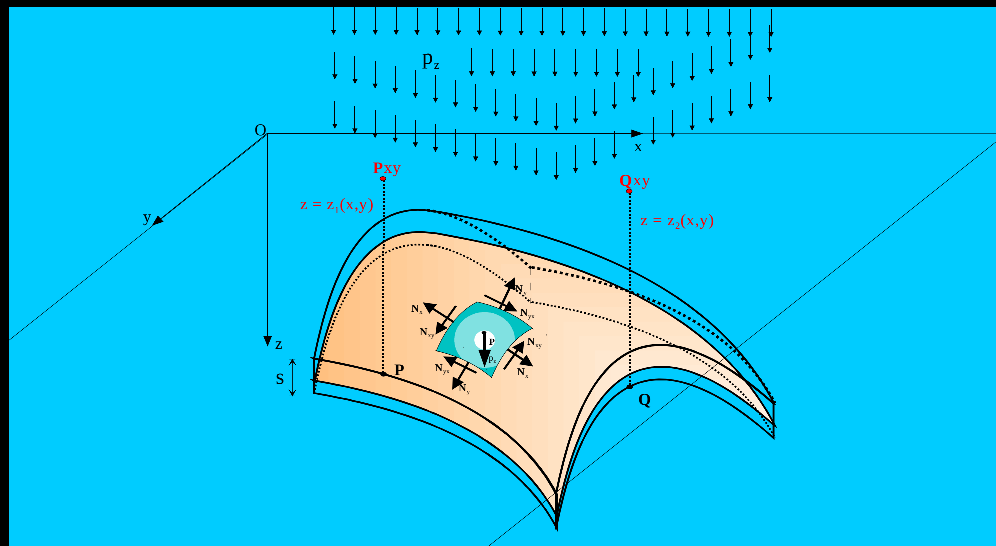



$$z_1(x,y) \le z(x,y) \le z_2(x,y)$$

$$N_x \le 0$$
$$N_y \le 0$$
$$N_{xy}^2 - N_x N_y \le 0$$

# Masonry vaults of general shape

Setting the problem in plane variables the equilibrium and admissibility conditions turn to




$$N_x = \bar{N}_x \frac{\cos\theta}{\cos\phi}$$

$$N_y = \bar{N}_y \frac{\cos\phi}{\cos\theta}$$

$$N_{xy} = \bar{N}_{xy}$$

$$p_x = \bar{p}_x \cos\theta \cos\phi$$

$$p_y = \bar{p}_y \cos\theta \cos\phi$$

$$\bar{N}_x = \frac{\partial^2\psi}{\partial y^2}$$

$$\bar{N}_y = \frac{\partial^2\psi}{\partial x^2}$$

$$\bar{N}_{xy} = \bar{N}_{yx} = -\frac{\partial^2\psi}{\partial x\,\partial y}.$$

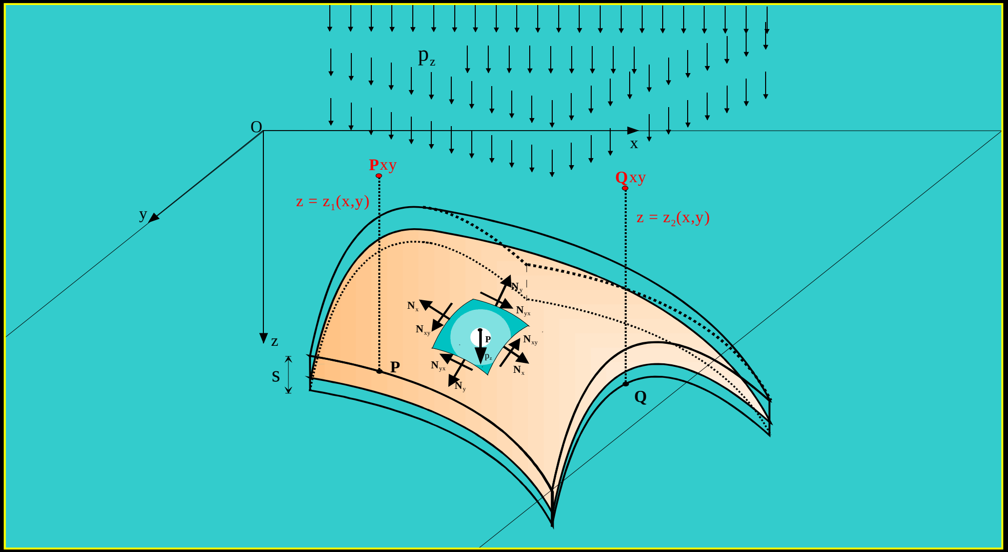
$$z_1(x,y) \le z(x,y) \le z_2(x,y)$$

$$\left.\begin{array}{l} N_x \le 0 \\ N_y \le 0 \\ N_{xy}^2 - N_x N_y \le 0 \end{array}\right\rangle \Leftrightarrow \left\langle\begin{array}{l} \bar{N}_x \le 0 \\ \bar{N}_y \le 0 \\ \bar{N}_{xy}^2 - \bar{N}_x \bar{N}_y \le 0 \end{array}\right.$$

$$\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2\psi}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2\psi}{\partial y^2}\frac{\partial^2 z}{\partial x^2} = -\bar{p}_z$$

# Masonry vaults of general shape

So, the key of the problem is how the stress function $\Psi(x,y)$ couples with the membrane function z(x,y)




It is interesting to note that if one takes $\Psi(x,y) = -k\, z(x,y)$

$$\overline{N}_x = -k\frac{\partial^2 z}{\partial y^2};\quad \overline{N}_y = -k\frac{\partial^2 z}{\partial x^2}$$

$$\overline{N}_{xy} = \overline{N}_{yx} = k\frac{\partial^2 z}{\partial x\, \partial y}.$$

$$z_1(x,y) \le z(x,y) \le z_2(x,y)$$

$$\left.\begin{array}{r} \overline{N}_x \le 0 \\ \overline{N}_y \le 0 \\ \overline{N}_{xy}^2 - \overline{N}_x\overline{N}_y \le 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \left.\begin{array}{l} \dfrac{\partial^2 z}{\partial y^2} \ge 0 \\ \dfrac{\partial^2 z}{\partial x^2} \ge 0 \end{array}\right\} \\ \dfrac{\partial^2 z}{\partial x^2}\dfrac{\partial^2 z}{\partial y^2} - \left[\dfrac{\partial^2 z}{\partial x \partial y}\right]^2 \ge 0 \end{array}\right.$$

z(x,y) is a convex function

**equilibrium**

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left[\frac{\partial^2 z}{\partial x\, \partial y}\right]^2 = \frac{\overline{p}_z}{2k} \ge 0$$

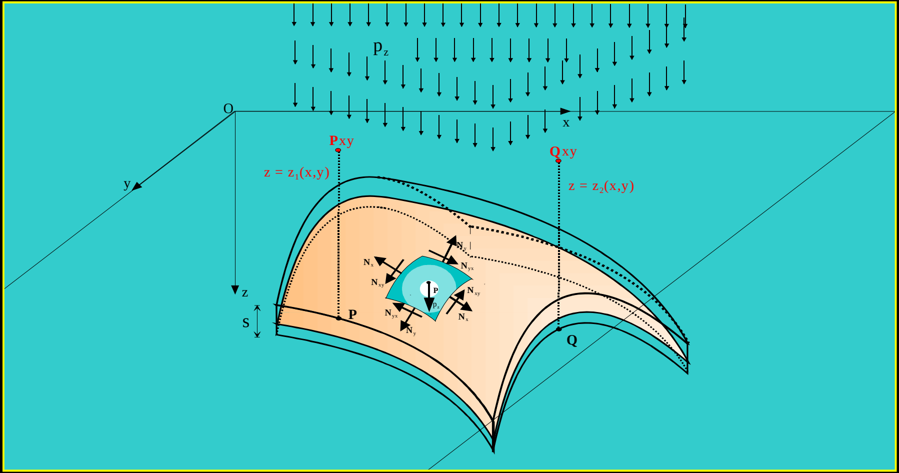
# Masonry vaults of general shape

In other words:

## **In a masonry vault equilibrium coincides with admissibility**

$$\overline{N}_x = -k\,\frac{\partial^2 z}{\partial y^2};\quad \overline{N}_y = -k\,\frac{\partial^2 z}{\partial x^2}$$

$$\overline{N}_{xy} = \overline{N}_{yx} = k\,\frac{\partial^2 z}{\partial x\,\partial y}.$$




$$\left.\begin{array}{r}\overline{N}_x \le 0\\ \overline{N}_y \le 0\\ \overline{N}_{xy}^2 - \overline{N}_x\overline{N}_y \le 0\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\left.\begin{array}{l}\dfrac{\partial^2 z}{\partial y^2} \ge 0\\ \dfrac{\partial^2 z}{\partial x^2} \ge 0\end{array}\right\}\\ \dfrac{\partial^2 z}{\partial x^2}\dfrac{\partial^2 z}{\partial y^2} - \left[\dfrac{\partial^2 z}{\partial x \partial y}\right]^2 \ge 0\end{array}\right.$$

z(x,y) is a concave function

$$z_1(x,y) \le z(x,y) \le z_2(x,y)$$

$$\frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2} - \left[\frac{\partial^2 z}{\partial x\,\partial y}\right]^2 = \frac{\overline{p}_z}{2k} \ge 0$$

# Masonry vaults of general shape

More in general: $\Psi(x,y) = F[z(x,y)]$

$$z_1(x,y) \le z(x,y) \le z_2(x,y)$$

$$\overline{N}_x = \frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial y}\right]^2 + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial y^2};\quad \overline{N}_y = \frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial x}\right]^2 + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x^2};\quad \overline{N}_{xy} = \overline{N}_{yx} = -\left(\frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x\,\partial y}\right).$$

$$\left(\frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial x}\right]^2 + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x^2}\right)\frac{\partial^2 z}{\partial y^2} - 2\left(\frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x\,\partial y}\right)\frac{\partial^2 z}{\partial x\,\partial y} + \left(\frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial y}\right]^2 + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial y^2}\right)\frac{\partial^2 z}{\partial x^2} = -\overline{p}_z$$

$$\left(\frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial x}\right]^2\frac{\partial^2 z}{\partial y^2} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}\right) - 2\left(\frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y}\right) + \left(\frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial y}\right]^2\frac{\partial^2 z}{\partial x^2} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial y^2}\frac{\partial^2 z}{\partial x^2}\right) = -\overline{p}_z$$

$$\frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial x}\right]^2\frac{\partial^2 z}{\partial y^2} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y} - 2\frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial y}\right]^2\frac{\partial^2 z}{\partial x^2} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial y^2}\frac{\partial^2 z}{\partial x^2} = -\overline{p}_z$$

$$\frac{\partial F}{\partial z}\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2}\frac{\partial^2 z}{\partial x^2}\right) + \frac{\partial^2 F}{\partial z^2}\left[\left(\frac{\partial z}{\partial x}\right)^2\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y} + \left(\frac{\partial z}{\partial y}\right)^2\frac{\partial^2 z}{\partial x^2}\right] = -\overline{p}_z$$

$$2\frac{\partial F}{\partial z}\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y}\right) + \frac{\partial^2 F}{\partial z^2}\left[\left(\frac{\partial z}{\partial x}\right)^2\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y} + \left(\frac{\partial z}{\partial y}\right)^2\frac{\partial^2 z}{\partial x^2}\right] = -\overline{p}_z$$

# Masonry vaults of general shape

More in general: $\Psi(x,y) = F[z(x,y)]$

$$z_1(x,y) \le z(x,y) \le z_2(x,y)$$

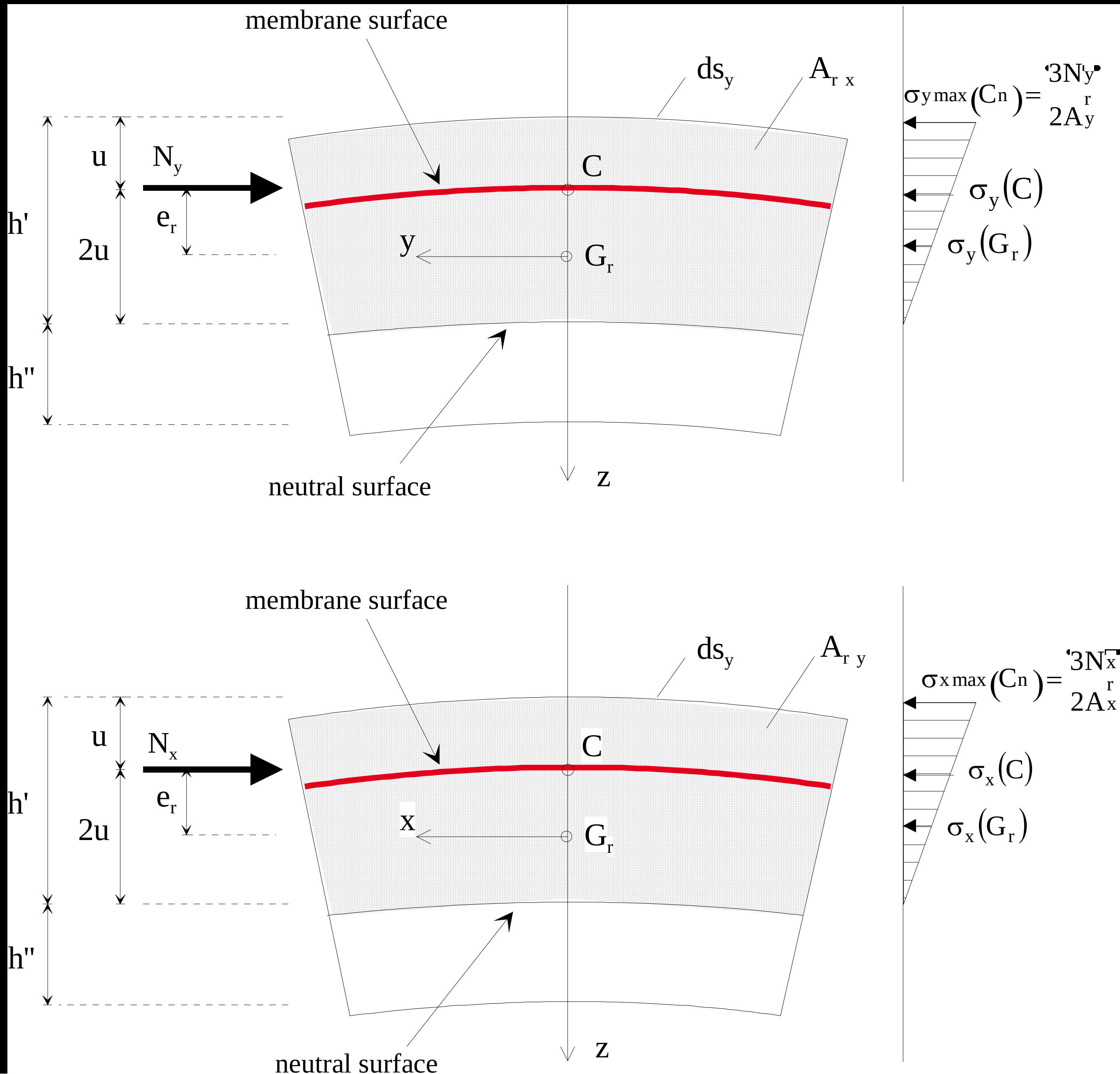
$$\overline{N}_x = \frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial y}\right]^2 + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial y^2};\quad \overline{N}_y = \frac{\partial^2 F}{\partial z^2}\left[\frac{\partial z}{\partial x}\right]^2 + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x^2};\quad \overline{N}_{xy} = \overline{N}_{yx} = -\left(\frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial^2 z}{\partial x\,\partial y}\right).$$

$$\left.\begin{array}{r}\overline{N}_x \le 0\\ \overline{N}_y \le 0\\ \overline{N}_{xy}^2 - \overline{N}_x\overline{N}_y \ge 0\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\left.\begin{array}{l}\dfrac{\partial^2 F}{\partial z^2}\left[\dfrac{\partial z}{\partial y}\right]^2 + \dfrac{\partial F}{\partial z}\dfrac{\partial^2 z}{\partial y^2} \le 0\\ \dfrac{\partial^2 F}{\partial z^2}\left[\dfrac{\partial z}{\partial x}\right]^2 + \dfrac{\partial F}{\partial z}\dfrac{\partial^2 z}{\partial x^2} \le 0\end{array}\right\}\\ \left(\dfrac{\partial^2 F}{\partial z^2}\left[\dfrac{\partial z}{\partial y}\right]^2 + \dfrac{\partial F}{\partial z}\dfrac{\partial^2 z}{\partial y^2}\right)\left(\dfrac{\partial^2 F}{\partial z^2}\left[\dfrac{\partial z}{\partial x}\right]^2 + \dfrac{\partial F}{\partial z}\dfrac{\partial^2 z}{\partial x^2}\right) - \left[\dfrac{\partial^2 F}{\partial z^2}\dfrac{\partial z}{\partial y}\dfrac{\partial z}{\partial x} + \dfrac{\partial F}{\partial z}\dfrac{\partial^2 z}{\partial x\,\partial y}\right]^2 \le 0\end{array}\right.$$

$$2\frac{\partial F}{\partial z}\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y}\right) + \frac{\partial^2 F}{\partial z^2}\left[\left(\frac{\partial z}{\partial x}\right)^2\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\,\partial y} + \left(\frac{\partial z}{\partial y}\right)^2\frac{\partial^2 z}{\partial x^2}\right] = -\overline{p}_z$$

# Masonry vaults of general shape

After membrane forces have been identified, stresses and consequently elasic energy are easily calculated



For instance, consider the following case

$$\psi(x,y) = -kz(x,y) + f(y) + g(x)$$

$$\Downarrow$$

$$\frac{\partial^2\psi}{\partial x^2} = -k\frac{\partial^2 z}{\partial x^2} + g''(x) \; ; \; \frac{\partial^2\psi}{\partial y^2} = -k\frac{\partial^2 z}{\partial y^2} + f''(y) \; ; \; \frac{\partial^2\psi}{\partial x\,\partial y} = -k\frac{\partial^2 z}{\partial x\,\partial y}$$

$$\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2\psi}{\partial x\,\partial y}\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2\psi}{\partial y^2}\frac{\partial^2 z}{\partial x^2} = -\overline{p}_z$$

$$\Downarrow$$

$$-2k\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} + 2k\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 + f''(y)\frac{\partial^2 z}{\partial x^2} + g''(x)\frac{\partial^2 z}{\partial y^2} = -\overline{p}_z$$

If one considers $\bar{p}_z = \bar{p}_z(x)\ ;\ z = z(x)$

one gets $f''(y)\dfrac{d^2 z(x)}{dx^2} = -\bar{p}_z(x) \Rightarrow f''(y) = -H = \text{const} = \overline{N}_x \le 0 \Rightarrow H \ge 0$

whence

$$z''(x) = \frac{\bar{p}_z(x)}{H} \ge 0\ ;\ \overline{N}_x = \frac{\partial^2\psi}{\partial y^2} = f''(y) = \text{const} \le 0\ ;\ \overline{N}_y = \frac{\partial^2\psi}{\partial x^2} = -kz''(x) + g''(x)\ ;\ \overline{N}_{xy} = \frac{\partial^2\psi}{\partial x\,\partial y} = 0$$

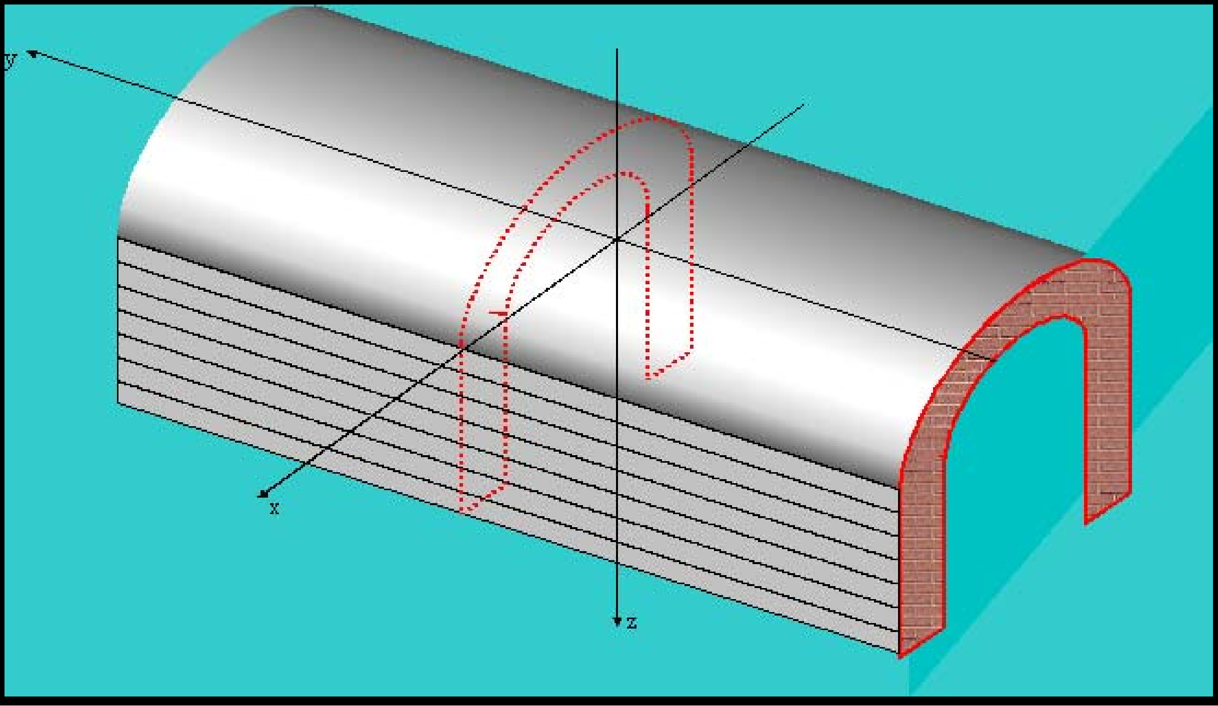
The solution is admissible if $-kz''(x) + g''(x) \le 0$

If one takes $-kz''(x) + g''(x) = 0$ the previous equations yield

$$z''(x) = \frac{\bar{p}_z(x)}{H} \ge 0$$

$$\overline{N}_x = \frac{\partial^2\psi}{\partial y^2} = f''(y) = \text{const} \le 0$$

$$\overline{N}_y = \frac{\partial^2\psi}{\partial x^2} = 0\ ;\ \overline{N}_{xy} = \frac{\partial^2\psi}{\partial x\,\partial y} = 0$$

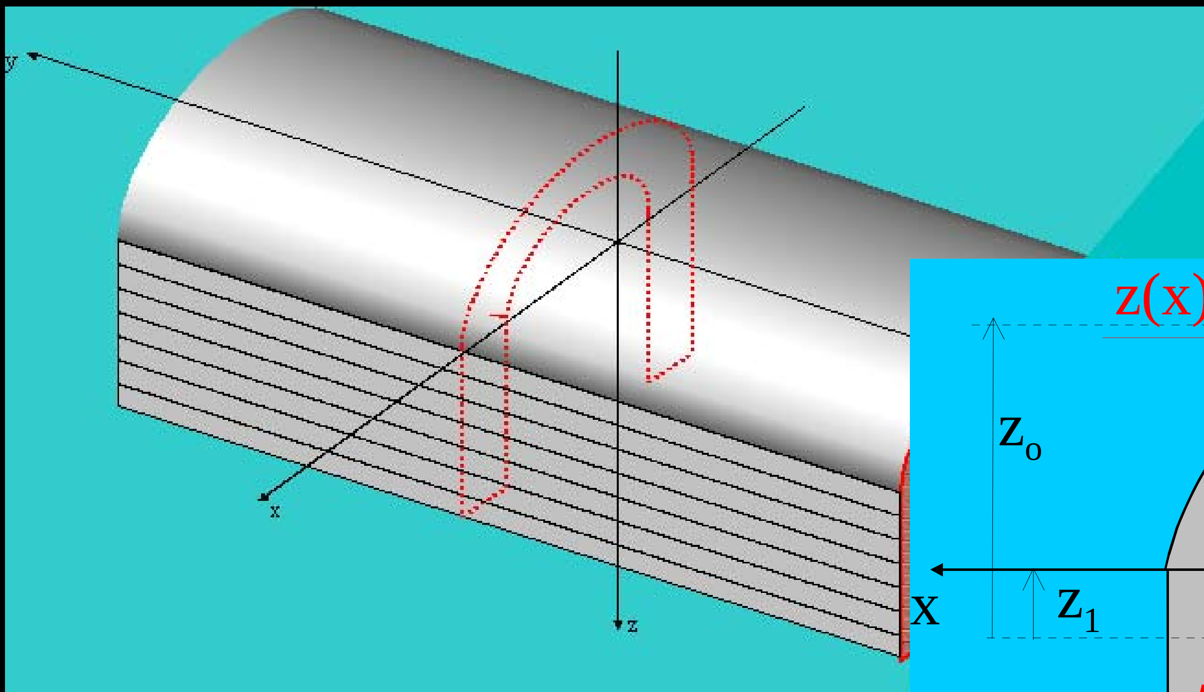


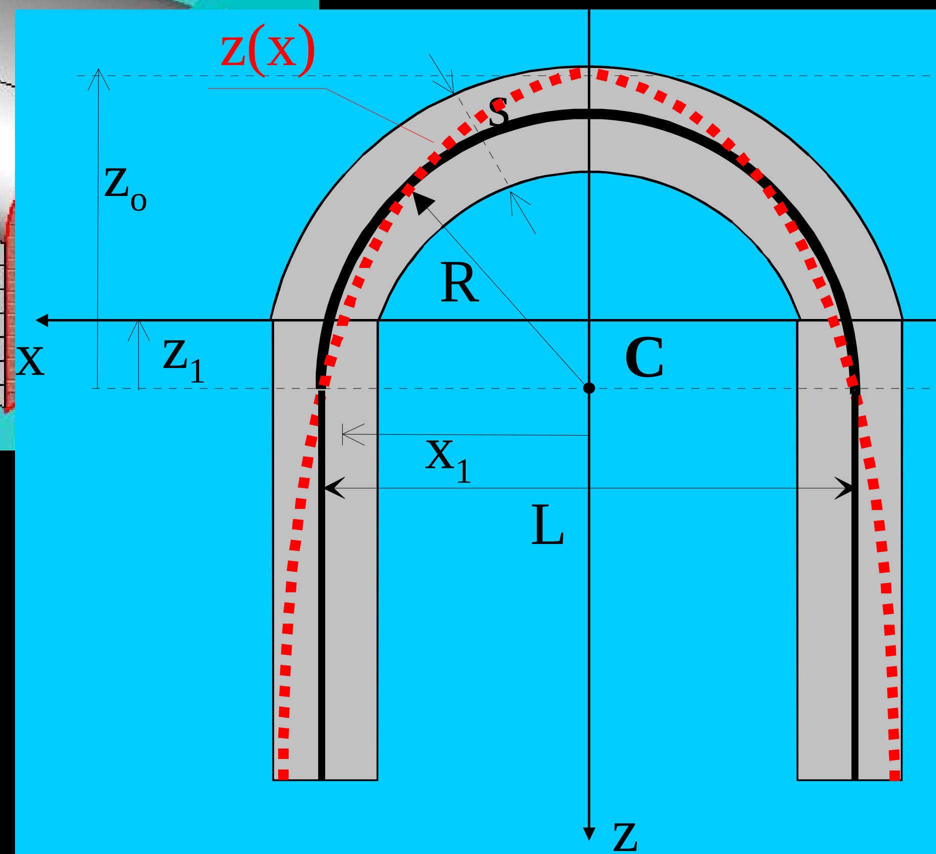

That is the well known solution for the barrel vault as a sequence of independent arches

The equation

$$z''(x) = \frac{\overline{p}_z(x)}{H} \geq 0$$

Is nothing else than the funicular line of the active load, and H the relevant thrust




That is the well known solution for the barrel vault as a sequence of independent arches

A second case can be obtained putting

$$\Psi(x,y) = ay^2\alpha(x)$$

$$z(x,y) = \frac{\Psi_o(x,y) - \Psi(x,y)}{2k}$$

with "L" a length parameter and $0 \le aL \le 1$

and $\alpha$(x) obeying the equation

$$q\alpha(x)\alpha''(x) - \alpha'^2(x) = k\bar{p}_z(x) \ge 0 \;;\; q = 1/2$$

Note that $\alpha$(x) is the function governing the internal forces in the membrane surface. In fact

$$\bar{N}_x(x,y) = \frac{\partial^2\Psi(x,y)}{\partial y^2} = 2a\alpha(x) \,;\, \bar{N}_{xy}(x,y) = \frac{\partial^2\Psi(x,y)}{\partial x\partial y} = 2ay\alpha'(x) \,;\, \bar{N}_y(x,y) = \frac{\partial^2\Psi(x,y)}{\partial x^2} = ay^2\alpha''(x)$$

Note that this equation, admits two distinct solutions, depending on the initial conditions.
Given the initial value $\alpha(-\lambda/2)$, and the absolute value of the corresponding derivative, one gets the cable equation with a positive (tension) force, or the arch equation with a negative (compression) force.
In fact. from the expression of $\Psi$(x,y), one gets $\bar{N}_x(x,y) = \frac{\partial^2\Psi(x,y)}{\partial y^2} = 2a\alpha(x)$

and if in the integration of the differential equation for $\alpha$(x) one starts with positive derivative, the value of $\alpha$(x) turns to be positive and so the second derivative

$$\alpha''(x) = \frac{k\bar{p}_z(x) + \alpha'^2(x)}{q\alpha(x)}$$

and the same happens for the normal force $N_x$.
The opposite happens if one starts with negative first derivative for $\alpha$(x).
Note that in the numerical solution, if one wants to start with $\alpha(-\lambda/2) = 0$ and any value of $\alpha'(-\lambda/2)$, the first step is in the form $\alpha(-L/2 + \Delta x) = \alpha'(-L/2)\Delta x$

$\Psi_o(x,y)$ is assigned in the form

$$\Psi_o(x,y) = G(x)y^2 + Q(x)$$

with G(x), Q(x) obeying the following differential equations

$$2a\alpha''(x)G(x) - 8a\alpha'(x)G'(x) + 2a\alpha(x)G''(x)y^2 = 8ka^2\overline{p}_z(x)$$
$$2a\alpha(x)Q''(x) = -2k\overline{p}_z(x)$$

The membrane surface remains identified by the equation:

$$z(x,y) = \frac{\Psi_o(x,y) - \Psi(x,y)}{2k} = \frac{G(x)y^2 + Q(x) - ay^2\alpha(x)}{2k}$$

The initial values of $\alpha(x)$, G(x), Q(x) and relevant derivatives, and the values of "a" and "k" remain undetermined and available to set additional constraints.

In fact, if the problem is symmetric in both directions <x> and <y>, the following conditions should be added:

$$z(x,y)=\frac{\Psi_o(x,y)-\Psi(x,y)}{2k}=\frac{G(x)y^2+Q(x)-ay^2\alpha(x)}{2k}$$

$$z(0,0)=\frac{Q(0)}{2k}=z_o\Rightarrow Q(0)=2kz_o$$

$$\left(\frac{\partial z}{\partial x}\right)_{x=0}=z_x(0,y)=\frac{G'(0)y^2+Q'(0)-ay^2\alpha'(0)}{2k}=0\ \ \forall y\in(-L/2,L/2)\Rightarrow\begin{cases}G'(0)-a\alpha'(0)=0\\Q'(0)=0\end{cases}$$

$$\left(\frac{\partial z}{\partial y}\right)_{y=0}=z_y(x,0)=\frac{2G(x)-2a\alpha(x)}{2k}y\Big|_{y=0}=0\ \Rightarrow\ \text{no condition is added.}$$

This feature is incorporated in the assumed solution

Still because of symmetry

$$\left(\frac{\partial z}{\partial x}\right)_{x=-L/2}=z_x\left(-\frac{L}{2},y\right)=\frac{G'(-L/2)y^2+Q'(-L/2)-ay^2\alpha'(-L/2)}{2k}=z_1(y)\le 0$$

$$\left(\frac{\partial z}{\partial x}\right)_{x=+L/2}=z_x\left(+\frac{L}{2},y\right)=\frac{G'(+L/2)y^2+Q'(+L/2)-ay^2\alpha'(+L/2)}{2k}=z_2(y)\ge 0$$

$$z_2(y)=-z_1(y)\ \ \forall y\Rightarrow\begin{cases}G'(+L/2)+G'(-L/2)=a[\alpha'(+L/2)+\alpha'(-L/2)]\\Q'(+L/2)=-Q'(-L/2)\end{cases}$$

Initial conditions for $\alpha(x)$, G(x) and Q(x) and the parameters "a" and "k" can be sought in order to meet the requirement that the membrane surface z = z(x,y) is in the interior of the vault thickness everywhere.

Note that the set functions satisfy the equations

$$\frac{\partial^2\Psi}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-2\frac{\partial^2\Psi}{\partial x\partial y}\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2\Psi}{\partial y^2}\frac{\partial^2 z}{\partial x^2}=-\bar{p}_z(x)$$

$$\mathbb{C}$$

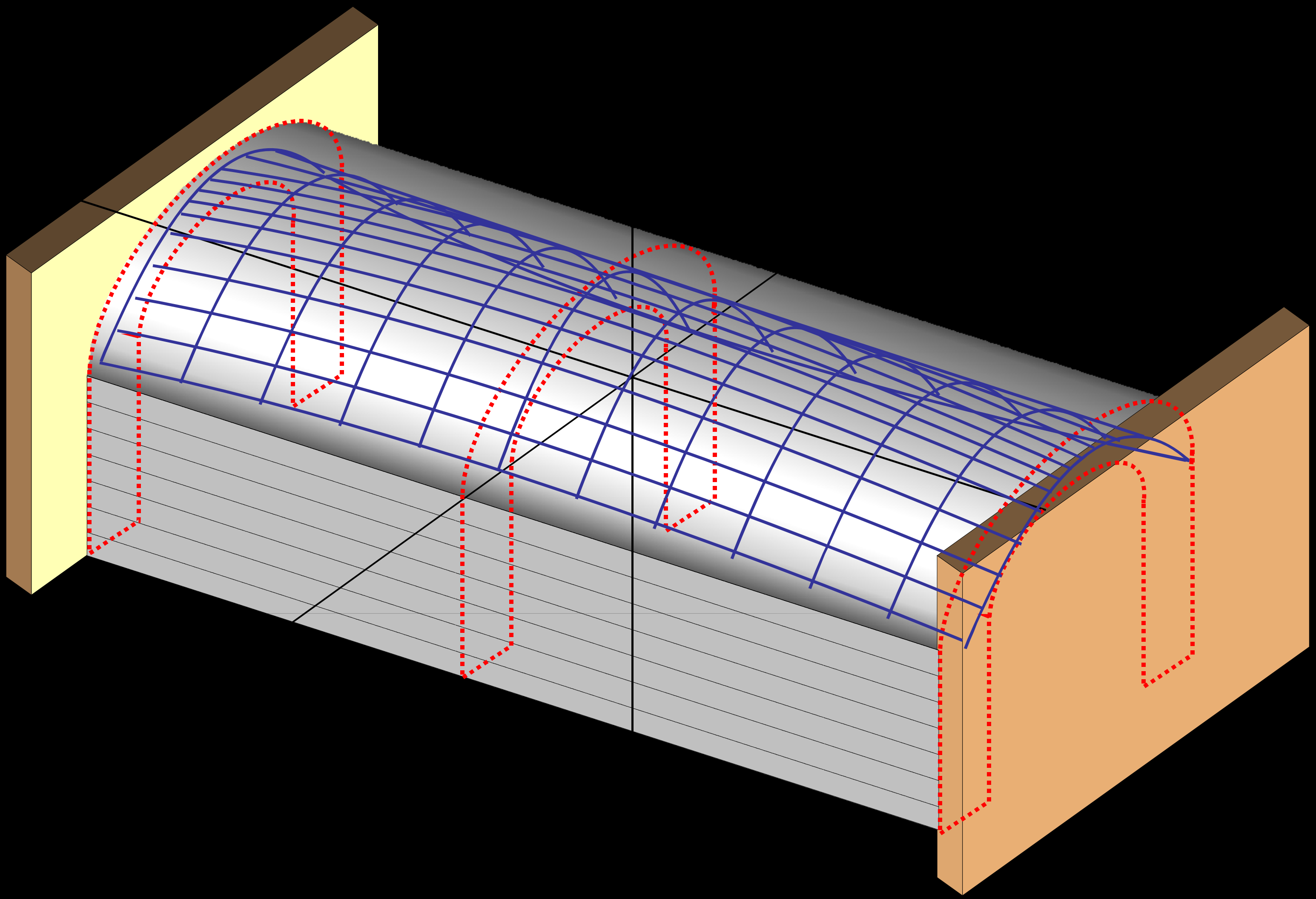
$$\frac{\partial^2\Psi}{\partial x^2}\frac{\partial^2\Psi}{\partial y^2}-\left(\frac{\partial^2\Psi}{\partial x\partial y}\right)^2=\rho(y)k\bar{p}_z(x)>0$$

$$\frac{\partial^2\Psi}{\partial x^2}\frac{\partial^2\Psi_o}{\partial y^2}-2\frac{\partial^2\Psi}{\partial x\partial y}\frac{\partial^2\Psi_o}{\partial x\partial y}+\frac{\partial^2\Psi}{\partial y^2}\frac{\partial^2\Psi_o}{\partial x^2}=-[1-\rho(y)]2k\bar{p}_z(x)$$

$$\text{with }\ 0\le\rho(y)\le 1\quad\forall y$$

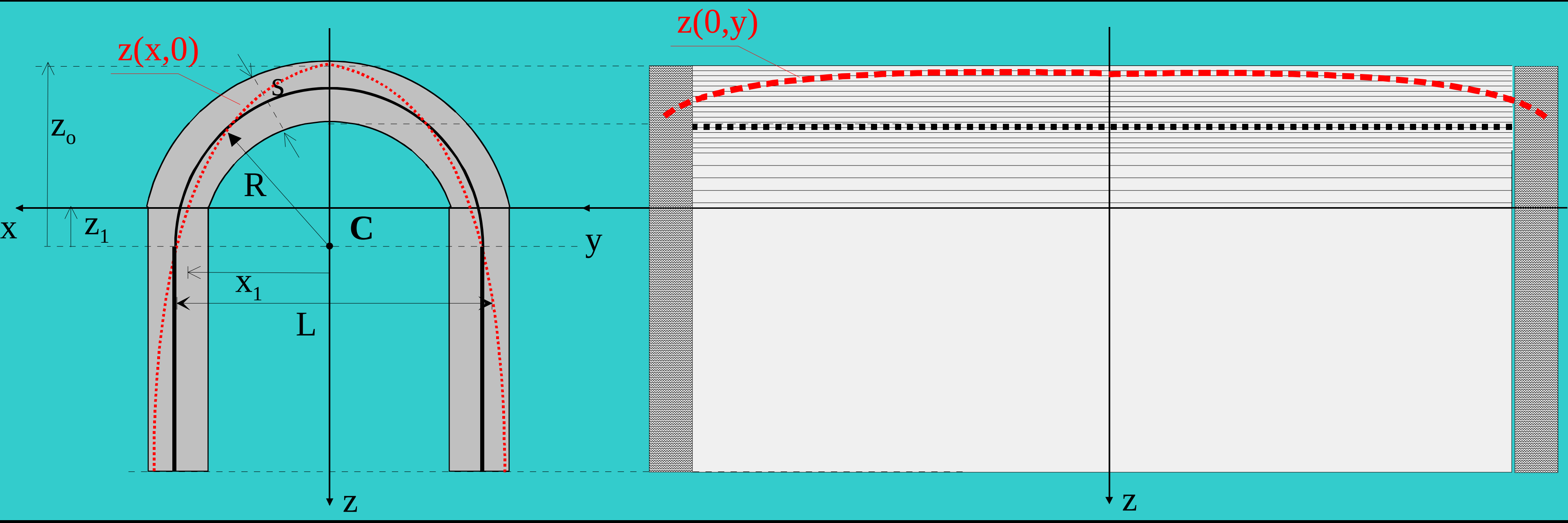
i.e. equilibrium and admissibility vs. vertical load are verified

If one looks at the z-surface resulting from the above positions



One realizes that this can be assumed as the membrane surface for a confined barrel vault

If one looks at the z-surface resultin from the above positions



# Other examples

- The parabolic vault under self-weight
- The parabolic vault under self-weight plus eccentric live load

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- The parabolic vault under self-weight
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$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left[\frac{\partial^2 z}{\partial x \, \partial y}\right]^2 = \frac{\overline{p}_z}{2k} \geq 0$$

(The Monge-Ampere Equation)

For admissible membrane surfaces, the thrust in solution is

$$Q^* = \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} H_{2Z}(x,y \mid z_{ij})\,\overline{p}_z(x,y)\,dydx}{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \left[H_{2Z}(x,y \mid z_{ij})\right]^2 dydx}$$

which tends to be positive as $H_{2z}(x,y) \rightarrow p_z(x,y)$.

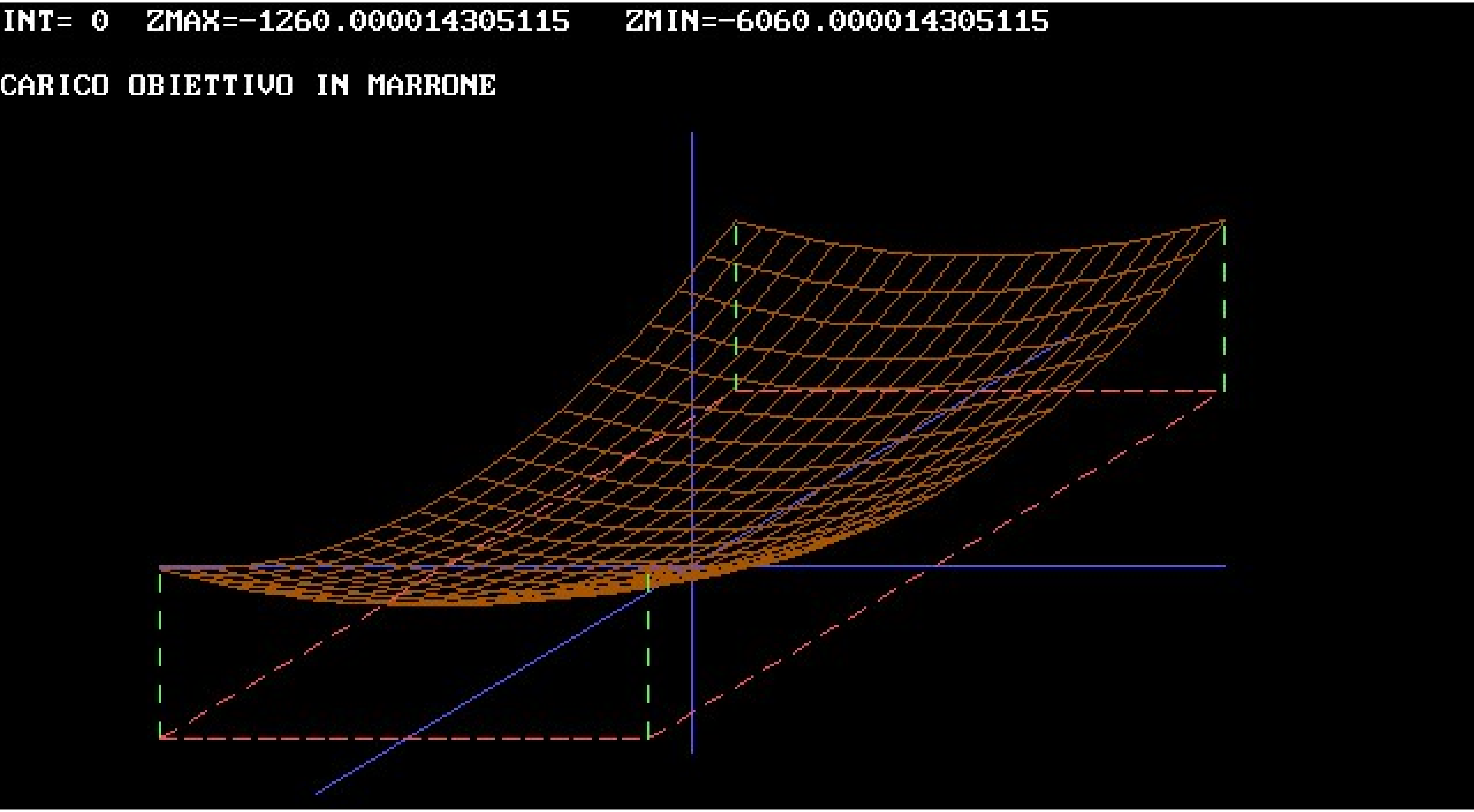
The m.s.d. function can then be reduced to its minimum value with respect to Q

$$E\left(z_{ij}, Q^*\right) = \min_{Q} E\left(z_{ij}, Q\right) =$$

$$= \frac{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \left[\overline{p}_z(x,y)\right]^2 dydx \int_{x_1}^{x_2}\int_{y_1}^{y_2} \left[H_{2Z}(x,y \mid z_{ij})\right]^2 dydx - \left[\int_{x_1}^{x_2}\int_{y_1}^{y_2} H_{2Z}(x,y \mid z_{ij})\,\overline{p}_z(x,y)\,dydx\right]^2}{\int_{x_1}^{x_2}\int_{y_1}^{y_2} \left[H_{2Z}(x,y \mid z_{ij})\right]^2 dydx}$$

which, by the Schwartz's inequality, is a non-negative definite function

Only permanent load



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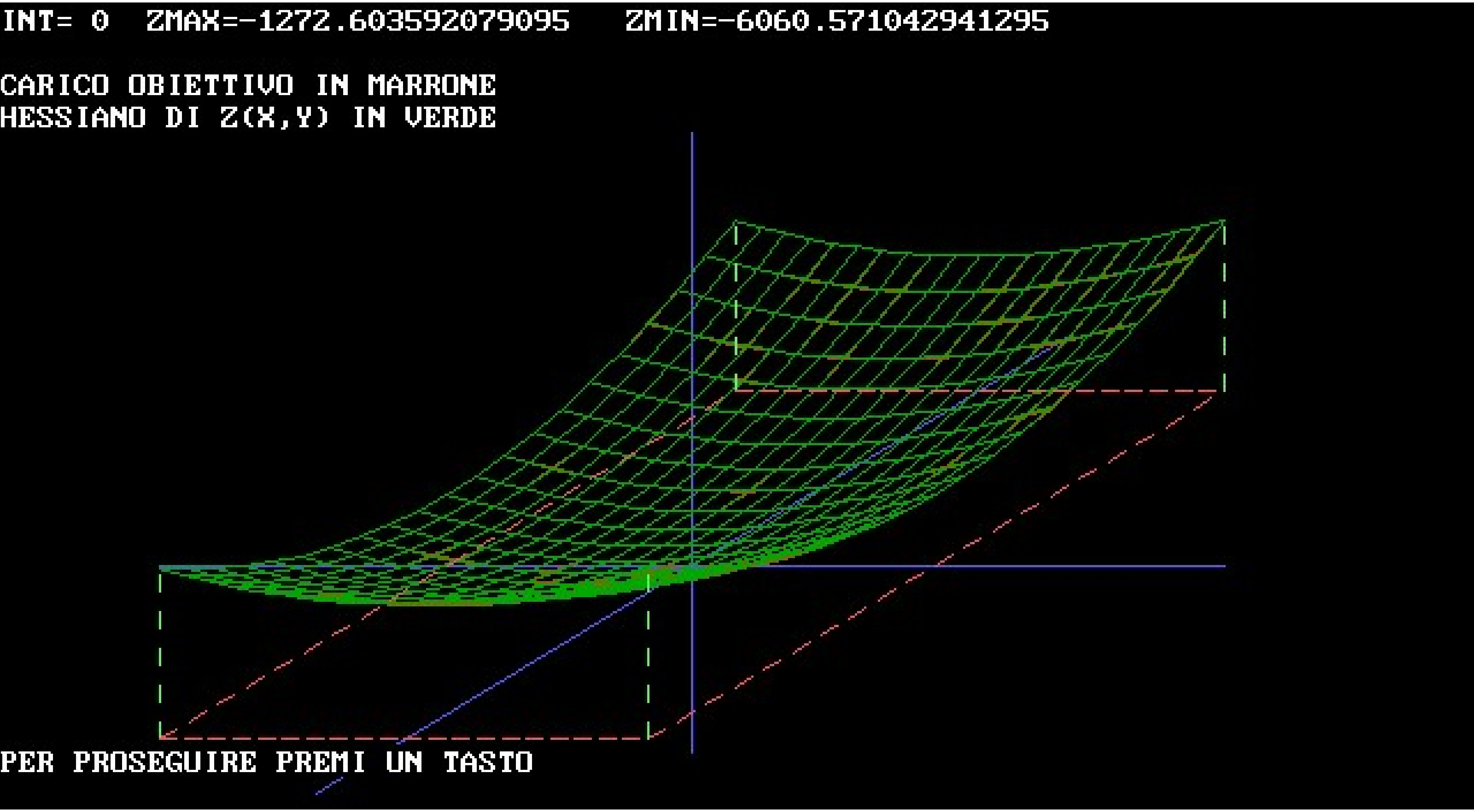

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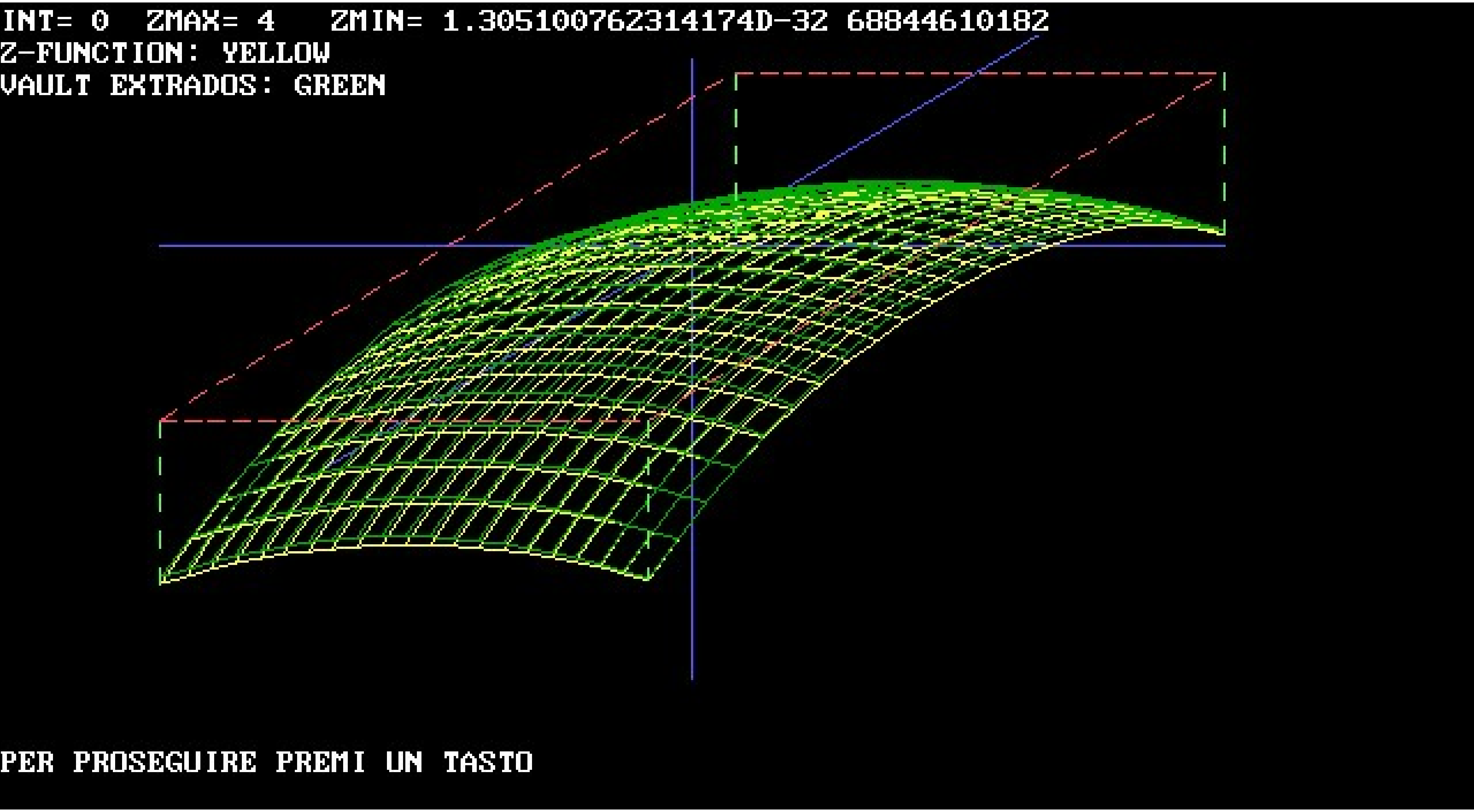

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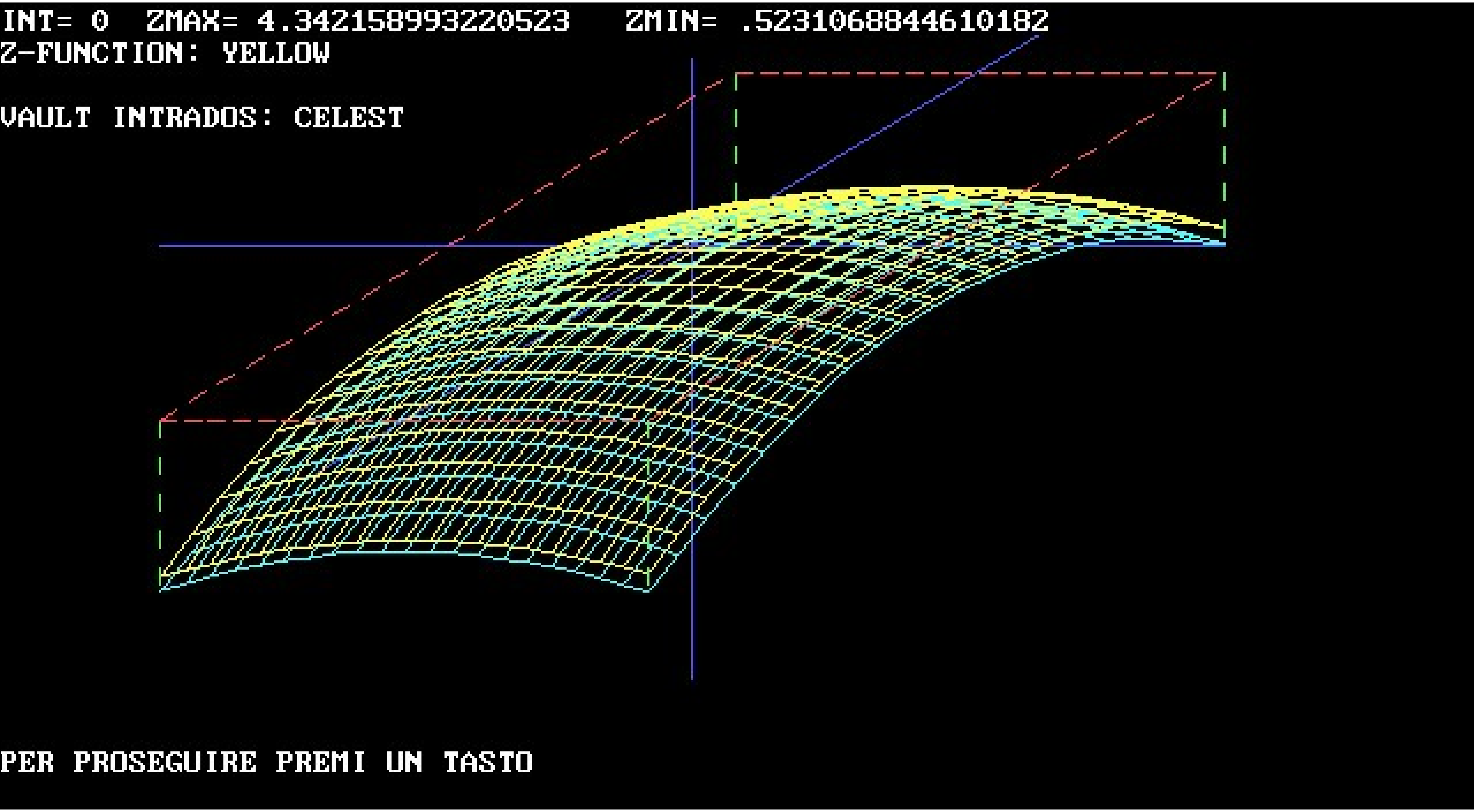

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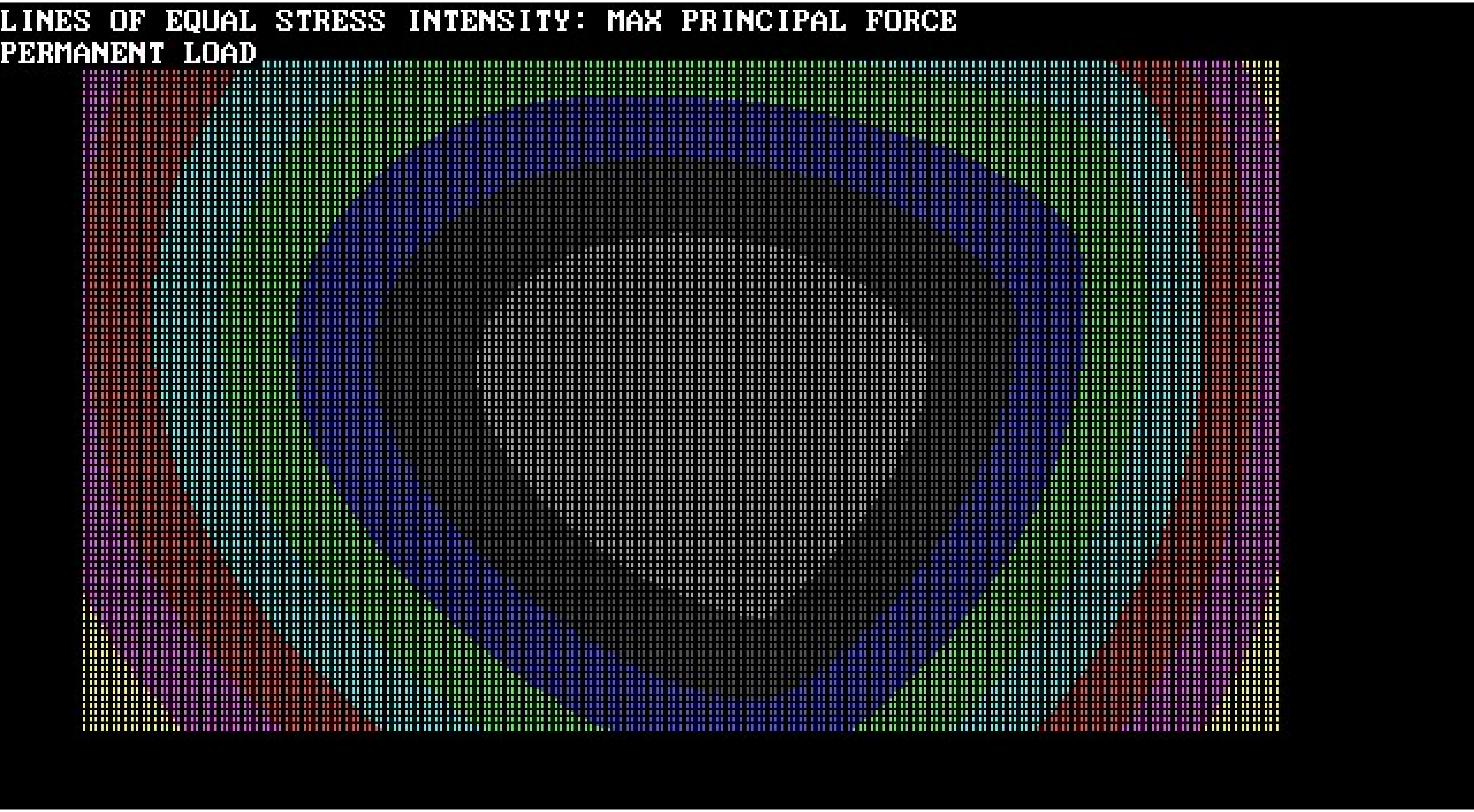

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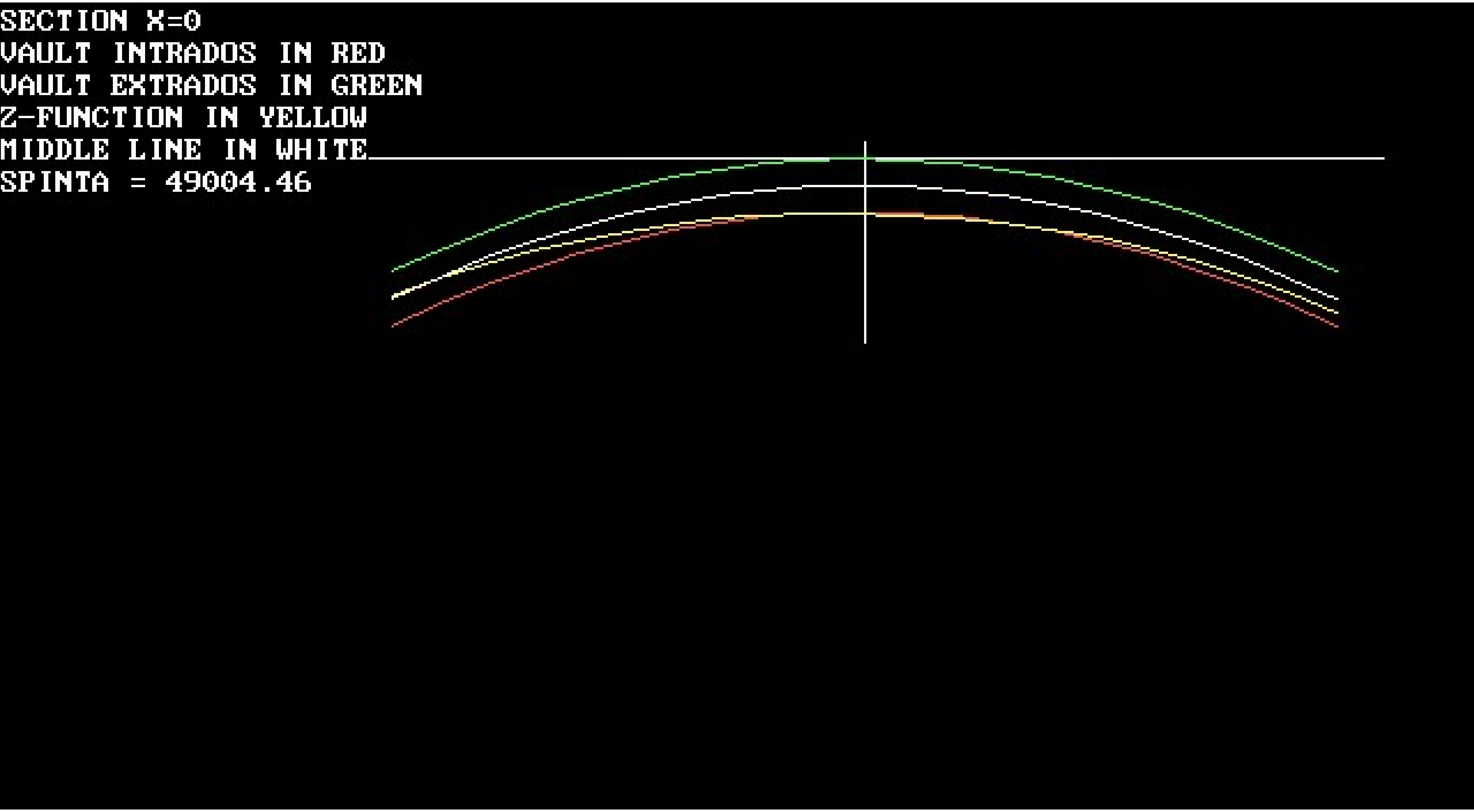

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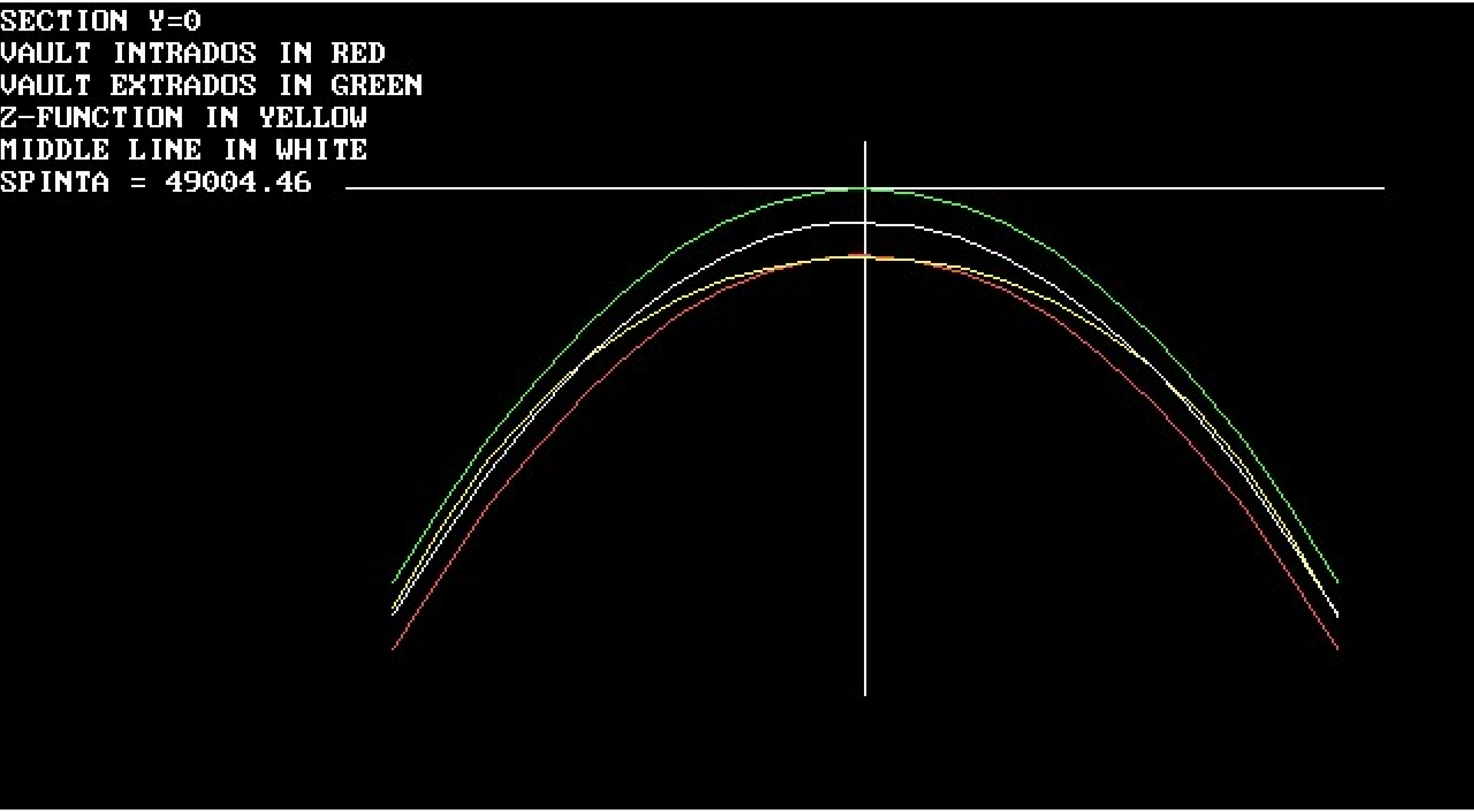

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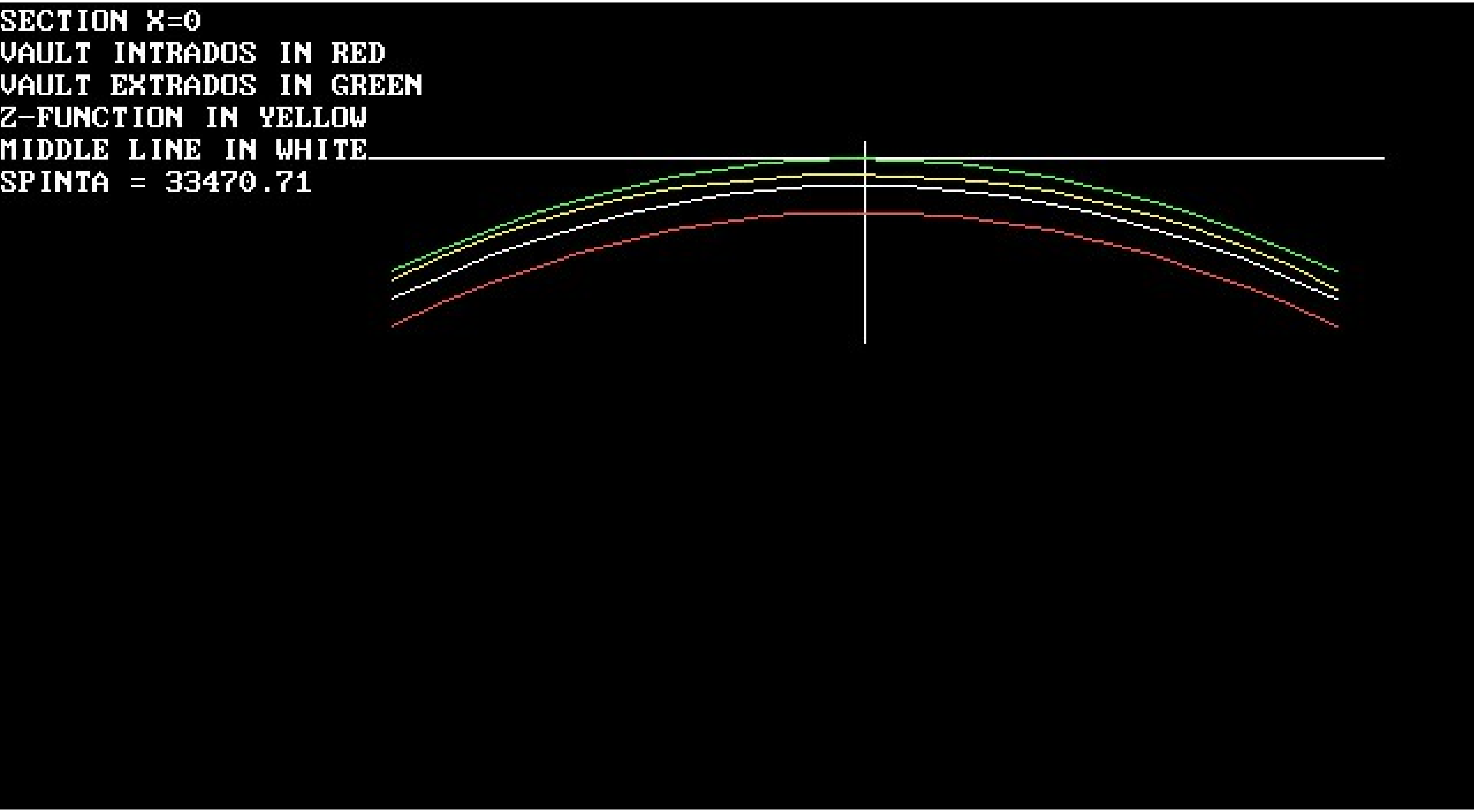

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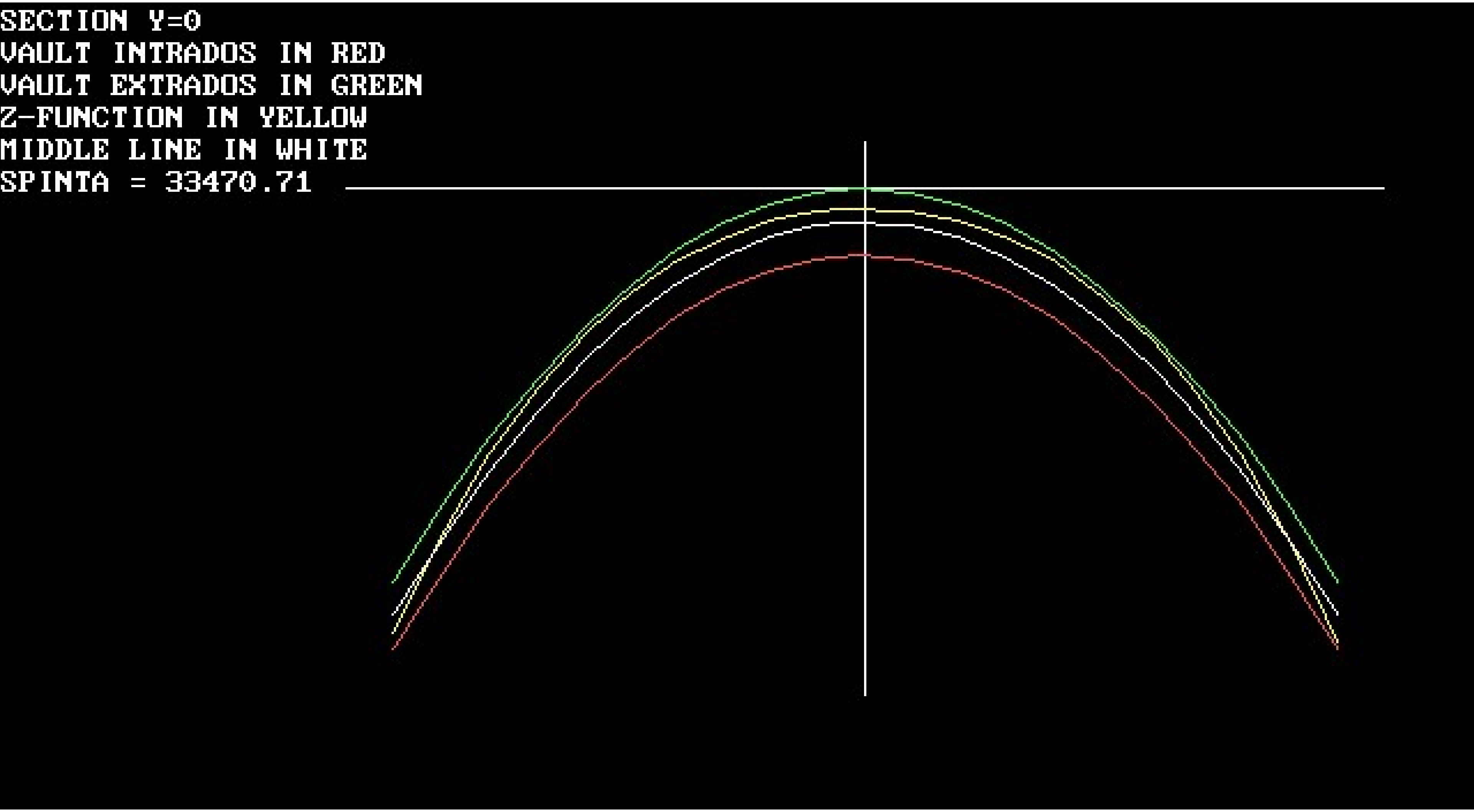

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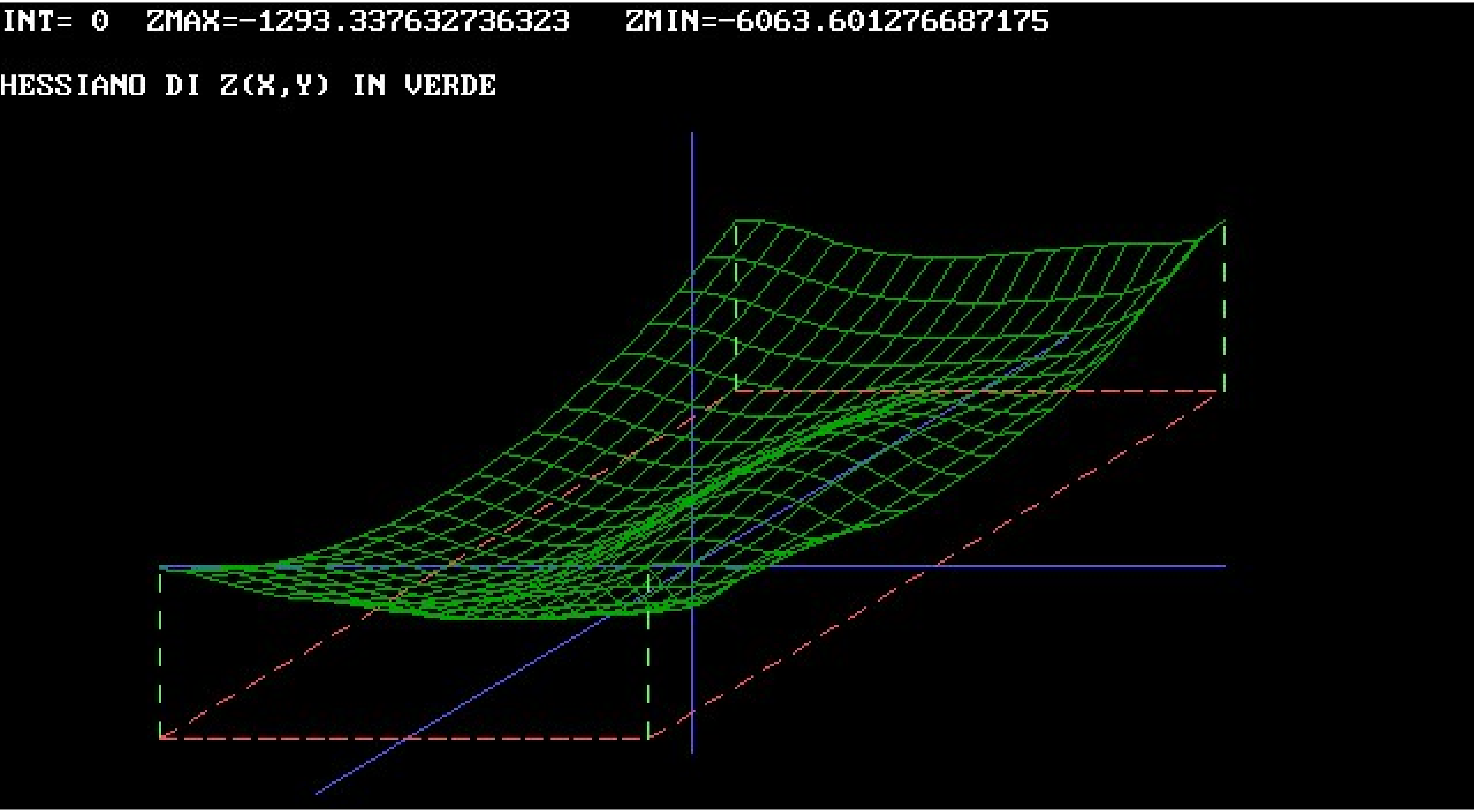

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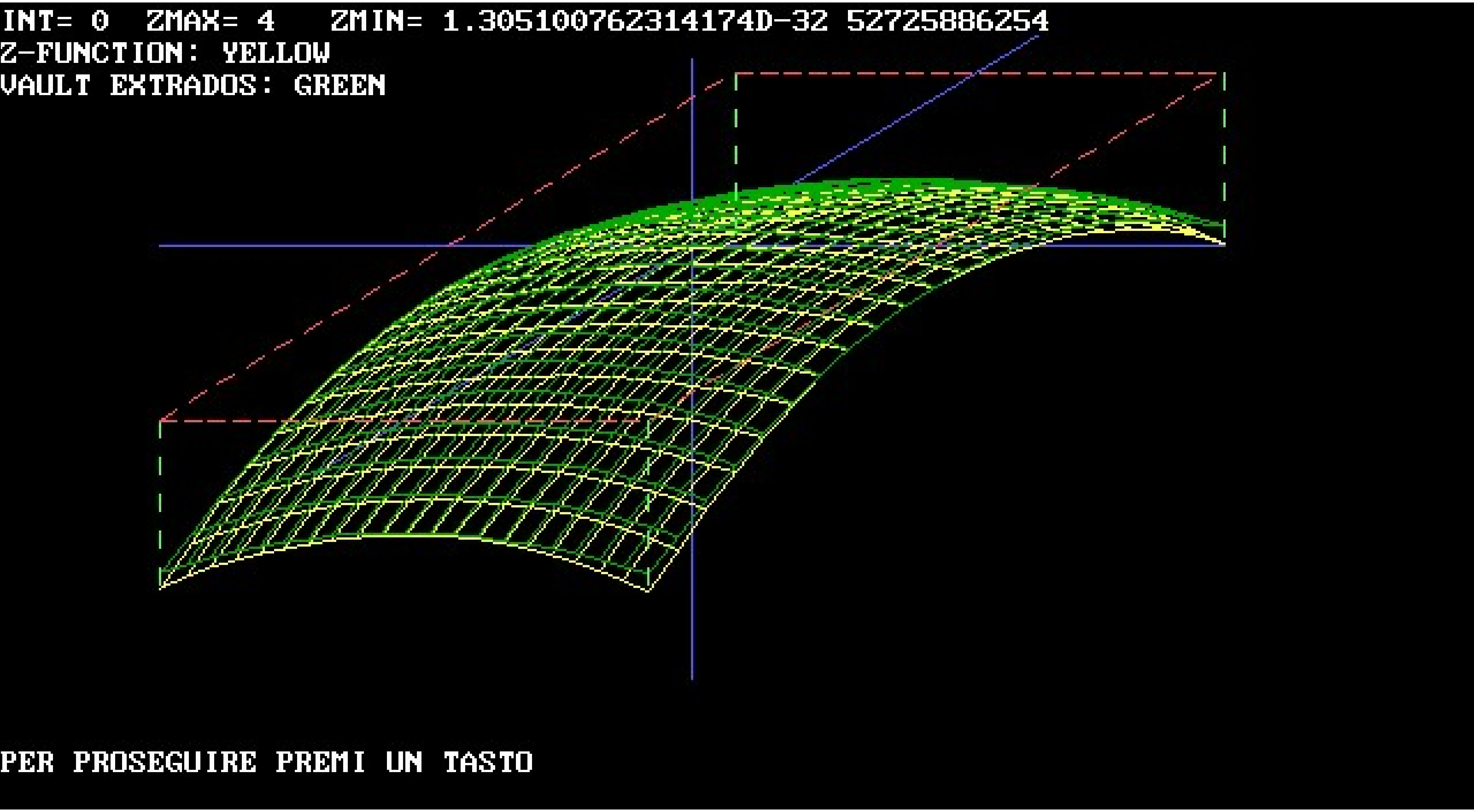

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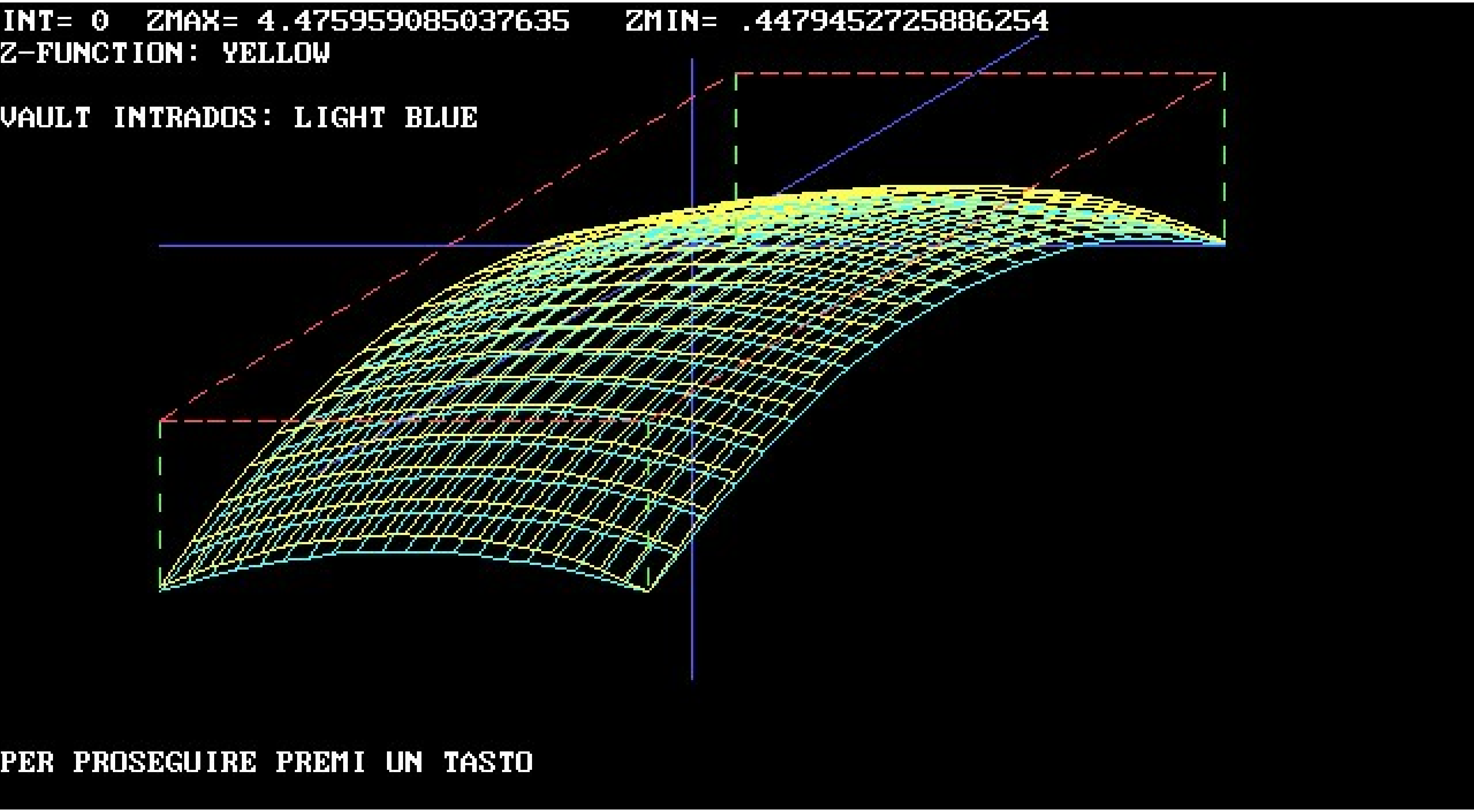

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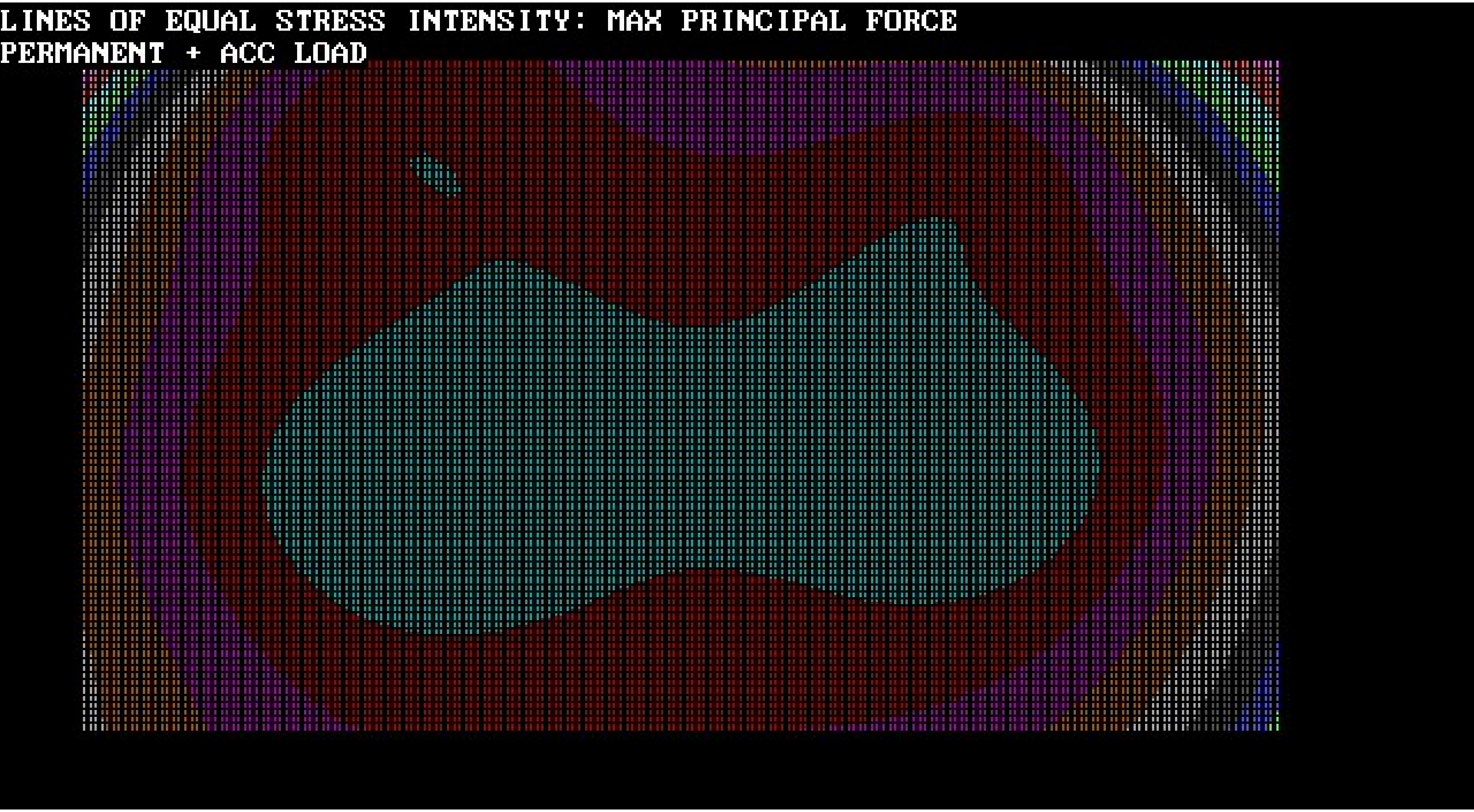

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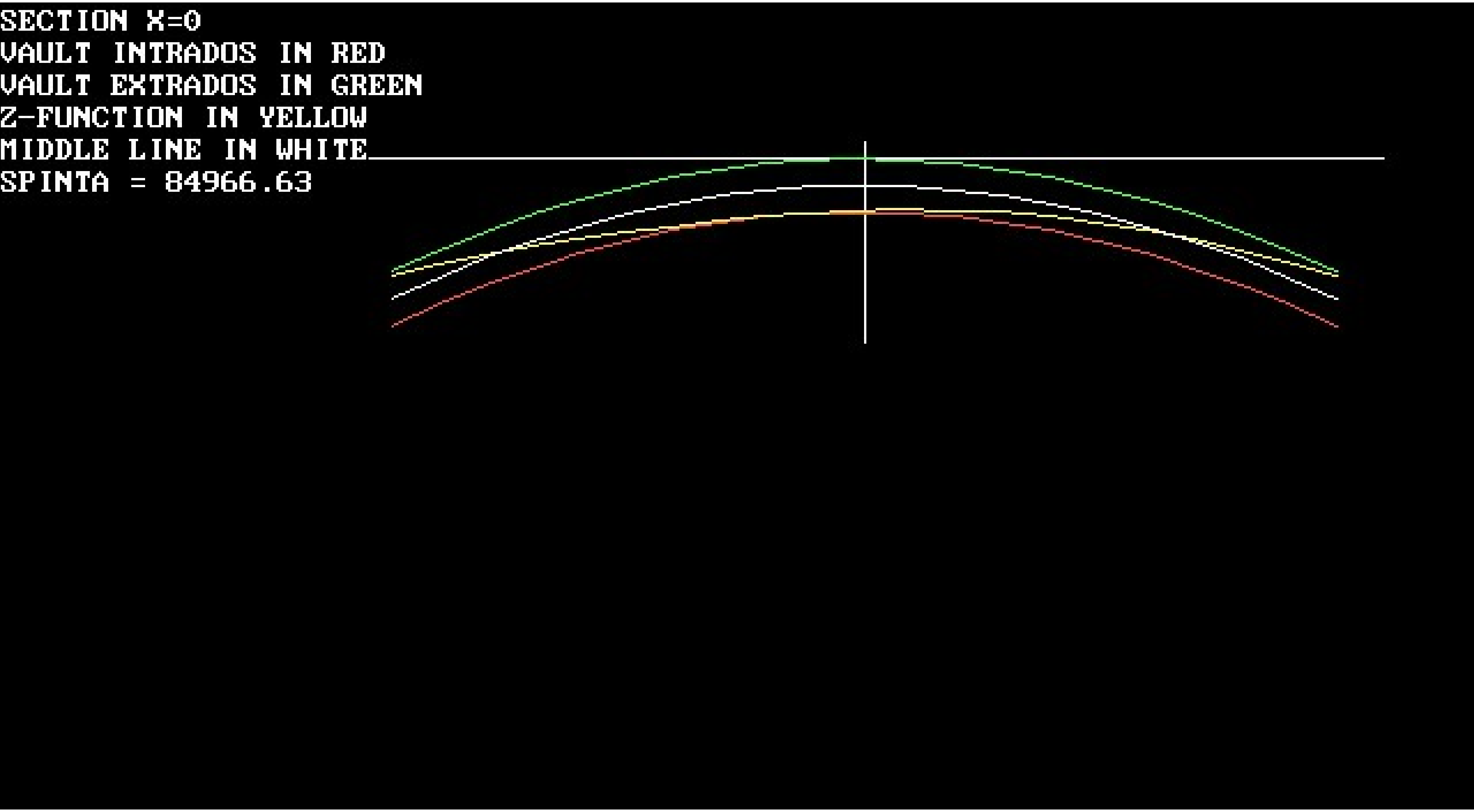

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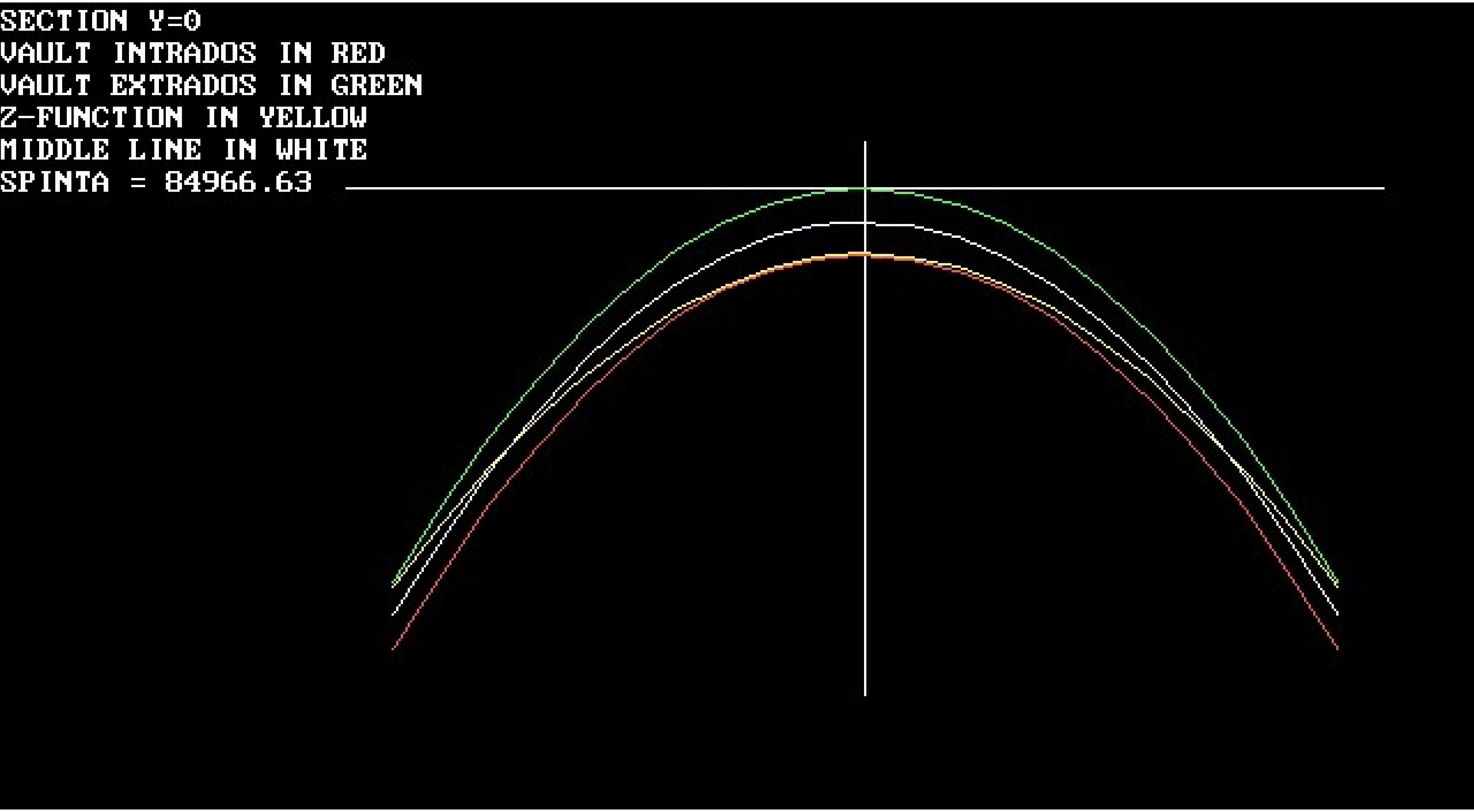

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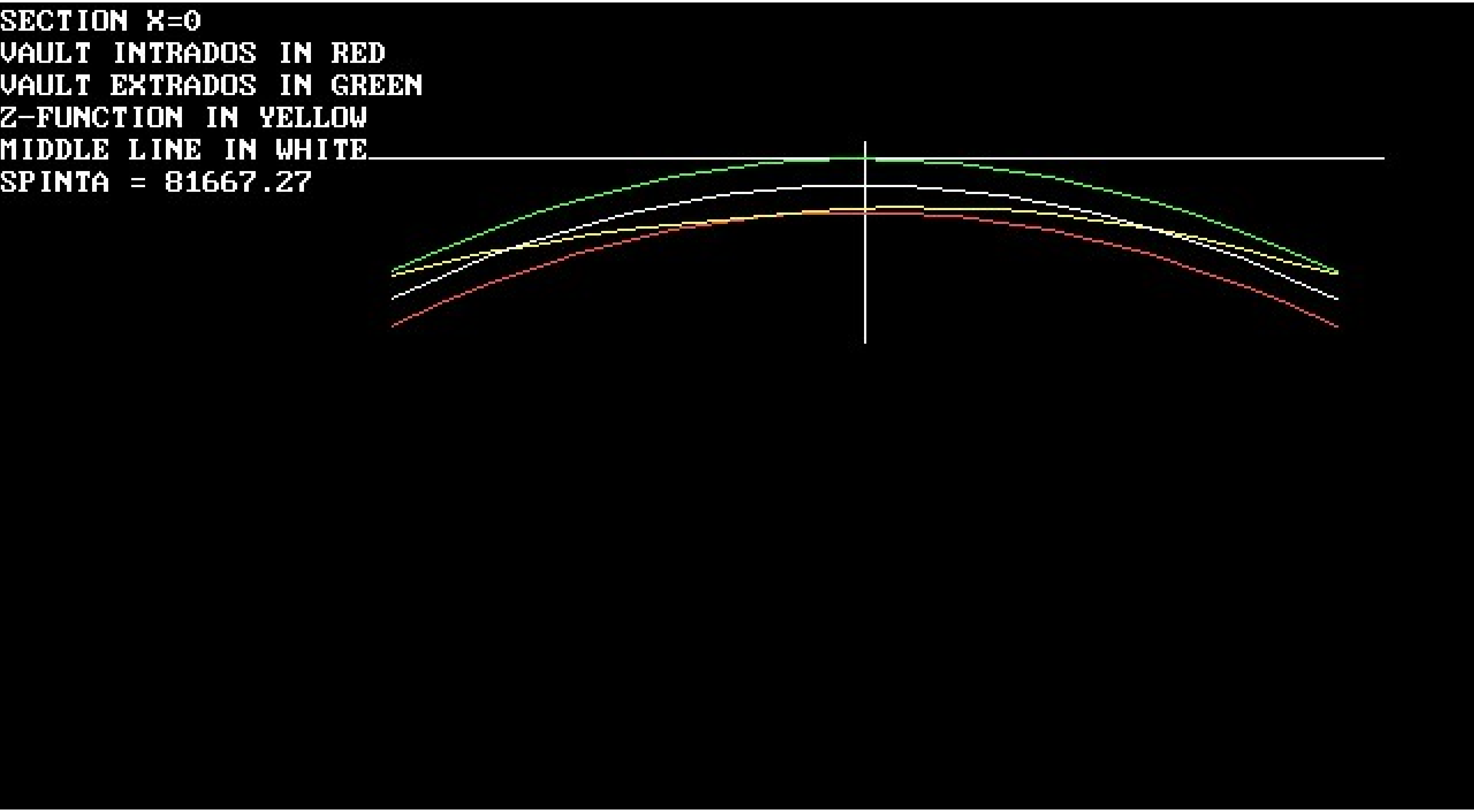

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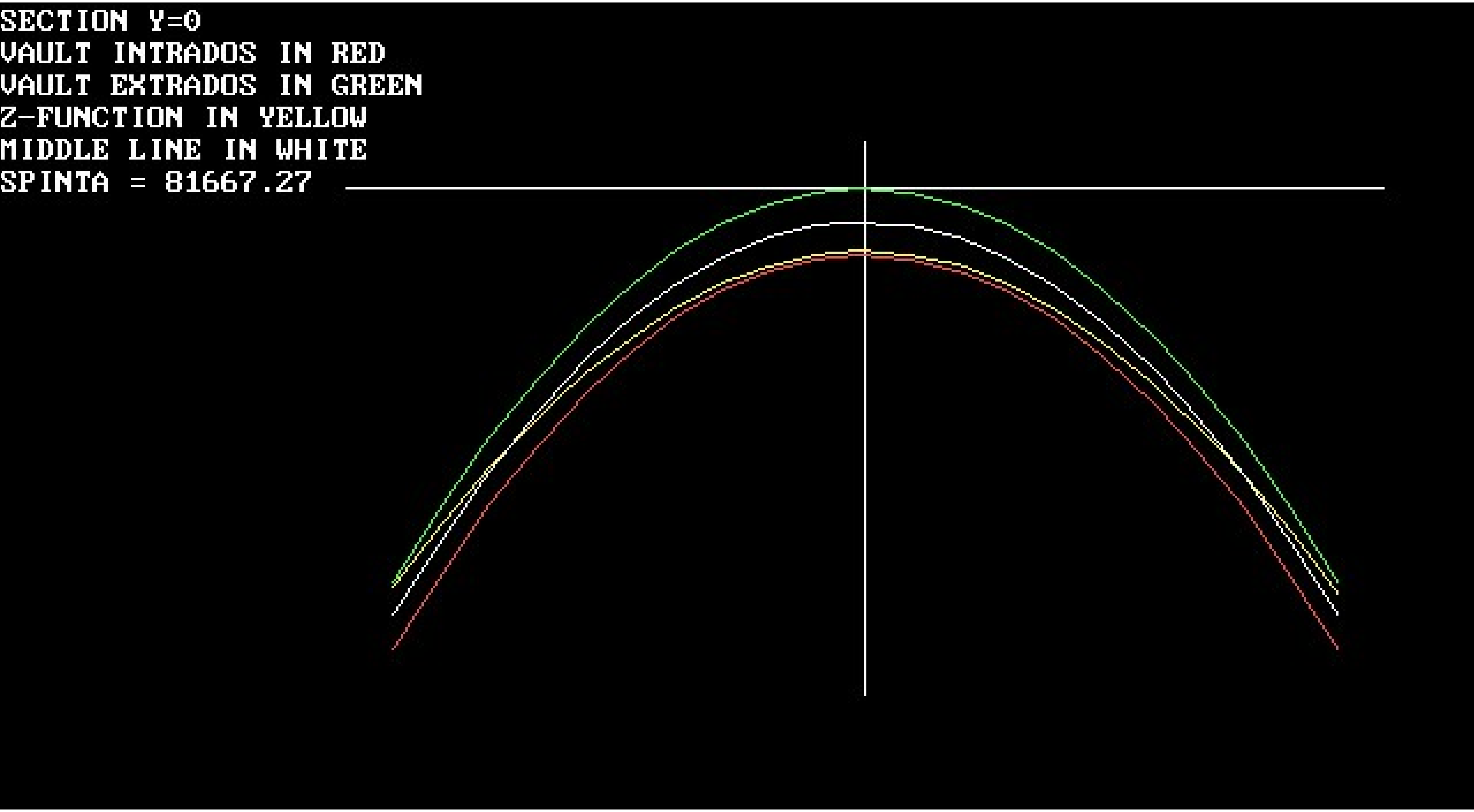

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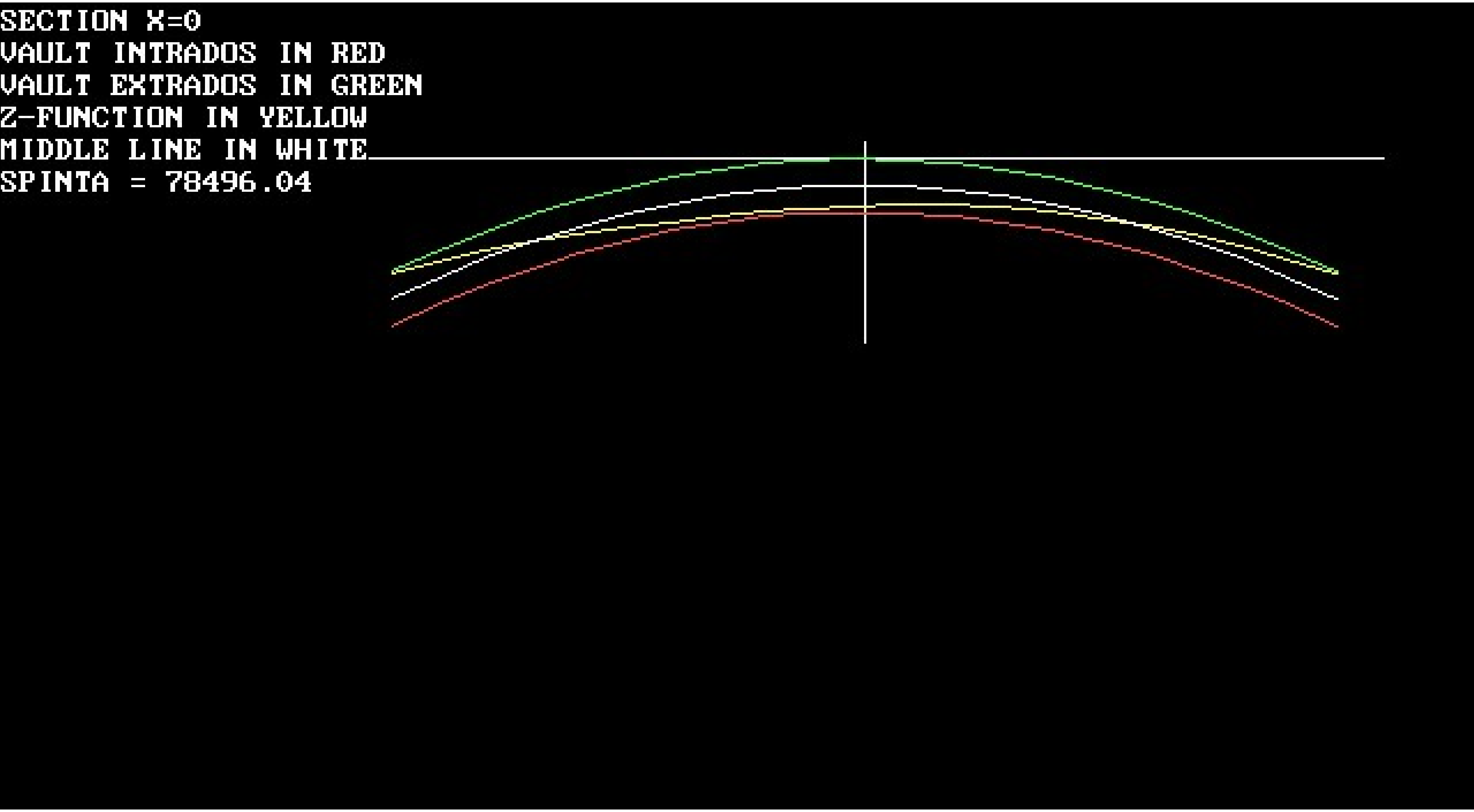

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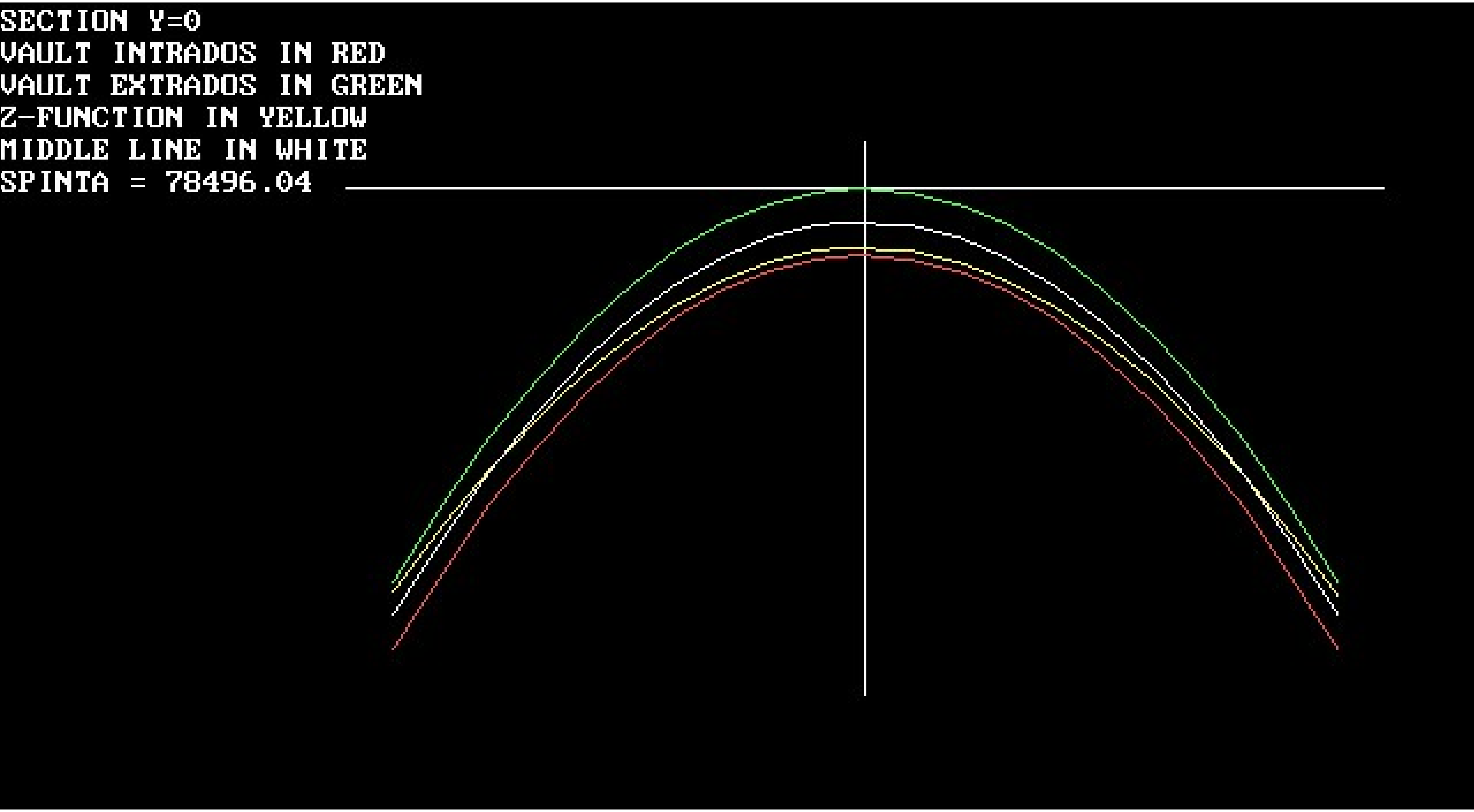

## CONCLUSIONS

An approach for the treatment of masonry vault analysis, based on the assumption that the material cannot resist tensile stresses is presented. After reducing the problem to a plane-stress problem, the stress function $\Psi(x,y)$ is introduced, as in the classical Pucher's approach.

For the vault to work as a no-tension structure, it is recognized that it is sufficient that a membrane surface completely included into the thickness of the vault exists, designed in way to resist applied loads by purely compressive membrane forces. **This idea was originally set by Heyman(1969), but it remained without significant developments up to the recent years.** In the paper It is proved that one possible solution under pure gravitational load can be searched by identifying the stress function with the membrane equation. In this case the following equation turns out, promptly including both equilibrium and admissibility

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left[\frac{\partial^2 z}{\partial x\, \partial y}\right]^2 = \frac{\bar{p}_z(x,y)}{2k} \geq 0$$

More in general, dependence of the stress function on the membrane equation can be postulated in a rather intuitive fashion, still leading, although in a less immediate way, to joint equations for equilibrium and admissibility.