Unilateral problems in structural analysis
Seventh meeting

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On uniqueness of solutions of frictional contact problems in linear elasticity

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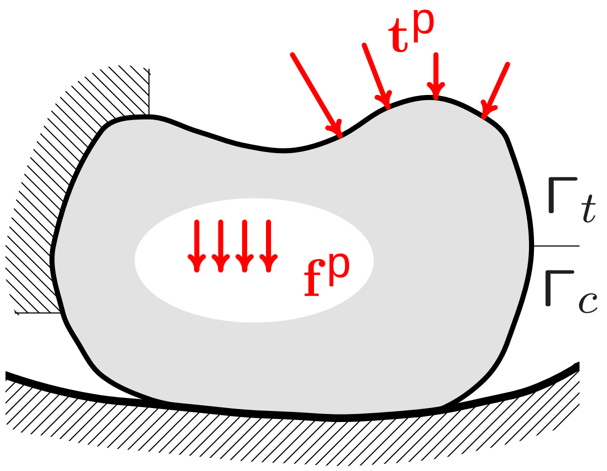
Elastic contact problems with Coulomb friction



Find a displacement field $\mathbf{u} : \Omega \mapsto \mathbb{R}^N$ such that:

$$\left|\begin{array}{ll} \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f}^{\mathsf{p}} = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}^{\mathsf{p}}, & \text{on } \Gamma_u, \\ \mathbf{t} \stackrel{\text{def}}{=} \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}^{\mathsf{p}}, & \text{on } \Gamma_t, \\ u_n - g^{\mathsf{p}} \leq 0, \quad t_n \leq 0, \quad \big(u_n - g^{\mathsf{p}}\big) t_n = 0, & \\ \text{and “tangential boundary conditions”}, & \text{on } \Gamma_c. \end{array}\right.$$

Tangential boundary conditions on Γ_c:

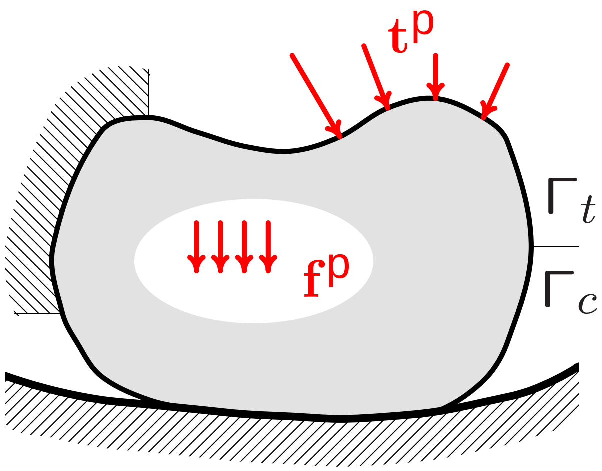
- Frictionless: $\mathbf{t}_t = \mathbf{0}$ (Signorini's problem),
- Coulomb friction: $\forall \mathbf{v}, \qquad \mathbf{t}_{\mathsf{t}} \cdot \big(\mathbf{v} - \dot{\mathbf{u}}_{\mathsf{t}}\big) - \mathcal{F} t_{\mathsf{n}} \big(|\mathbf{v}| - |\dot{\mathbf{u}}_{\mathsf{t}}|\big) \geq 0,$
- “static” Coulomb friction: $\forall \mathbf{v}, \qquad \mathbf{t}_{\mathsf{t}} \cdot \big(\mathbf{v} - \mathbf{u}_{\mathsf{t}}\big) - \mathcal{F} t_{\mathsf{n}} \big(|\mathbf{v}| - |\mathbf{u}_{\mathsf{t}}|\big) \geq 0,$
- “static” Tresca friction: $\forall \mathbf{v}, \qquad \mathbf{t}_{\mathsf{t}} \cdot \big(\mathbf{v} - \mathbf{u}_{\mathsf{t}}\big) + \mathcal{S} \big(|\mathbf{v}| - |\mathbf{u}_{\mathsf{t}}|\big) \geq 0.$

$\longrightarrow$ **Fixed point** strategy for “static” Coulomb friction (Duvaut & Lions, 70).

Analysis of this problem : what is known and what is not known yet.

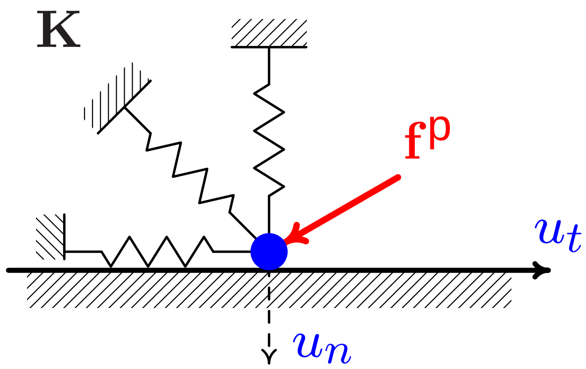
Continuous (static) problem

- Existence for small $\mathcal{F} < \mathcal{F}_c$ (Tikhonov fixed point, JARUŠEK, 1983).
- Example of multiple solutions (non-uniqueness) for large values of $\mathcal{F}$ (HILD, 2004).
- **Nothing** is known about uniqueness for arbitrarily small $\mathcal{F}$!



Discrete counterpart or discretized (static) problem

- Existence for all $\mathcal{F} \in \mathbb{R}^+$ (Brouwer fixed point).
- Uniqueness for small $\mathcal{F} < \mathcal{F}_c$ (Banach fixed point).
- Example of multiple solutions (non-uniqueness) for large values of $\mathcal{F}$ (KLARBRING, 1990).

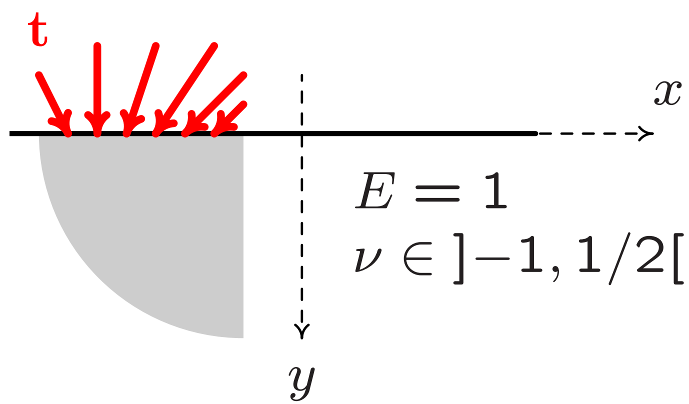


A particular geometry: the 2D elastic half-space

The Neumann-Dirichlet operator is known explicitly:

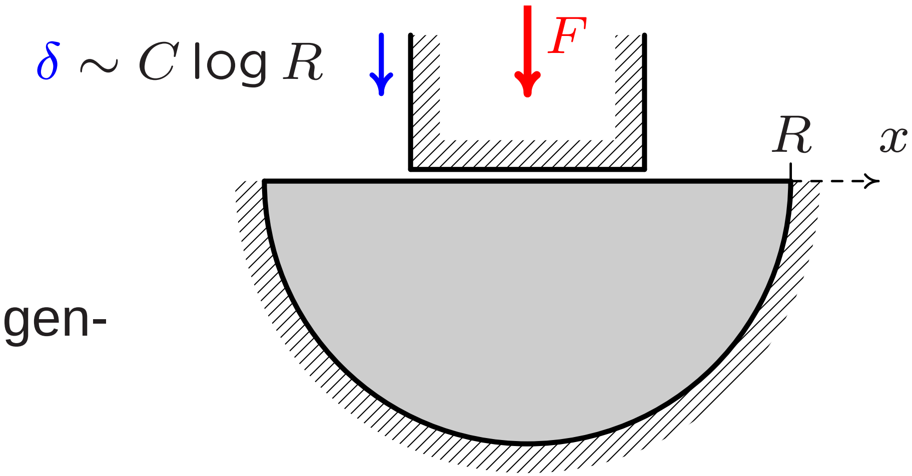
$$\frac{1}{2(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_x = \frac{1}{\pi}\oint_{-1}^{1}\frac{t_x(t)}{t-x}\,\mathrm{d}t - \frac{1-2\nu}{2(1-\nu)}t_y(x),$$

$$\frac{1}{2(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_y = \frac{1}{\pi}\oint_{-1}^{1}\frac{t_y(t)}{t-x}\,\mathrm{d}t + \frac{1-2\nu}{2(1-\nu)}t_x(x).$$



But, there are technical difficulties:

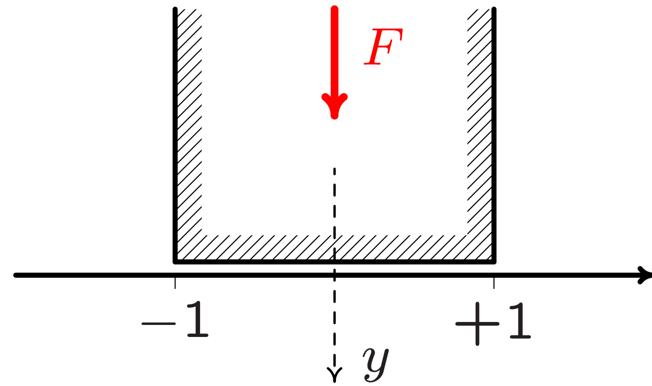
- the displacement is infinite at infinity,
- the solutions have unbounded energy, in general.



The Signorini problem for the 2D elastic half-space

$$\frac{1}{2(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_x = \frac{1}{\pi}\oint_{-1}^{1}\frac{t_x(t)}{t-x}\,\mathrm{d}t - \frac{1-2\nu}{2(1-\nu)}\,t_y(x),$$

$$\frac{1}{2(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_y = \frac{1}{\pi}\oint_{-1}^{1}\frac{t_y(t)}{t-x}\,\mathrm{d}t + \frac{1-2\nu}{2(1-\nu)}\,t_x(x).$$



$$\frac{1}{\pi}\oint_{-1}^{1}\frac{t_y(t)}{t-x}\,\mathrm{d}t = 0$$

$$\Downarrow$$

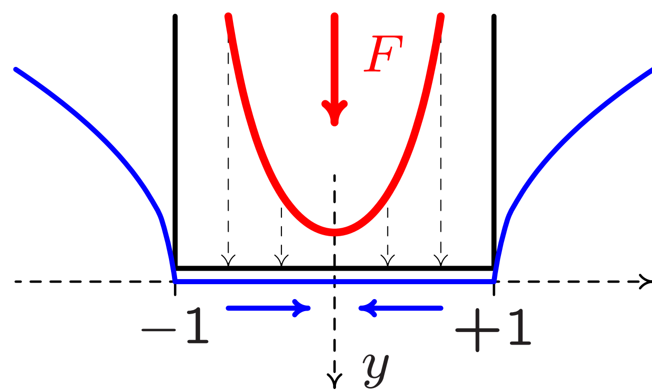
$$t_y(x) = \frac{F}{\pi\sqrt{1-x^2}}$$

The problem can be put under the form of a variational inequality by use of the bilinear form:

$$a(t_1, t_2) = -\int_{-1}^{1}\int_{-1}^{1} t_1(x)\, t_2(t)\ \log|x-t|\,\mathrm{d}x\,\mathrm{d}t,$$

$$= -\left\langle t_1, t_2 * \log|\cdot|\right\rangle_{H^{-1/2},H^{1/2}},$$

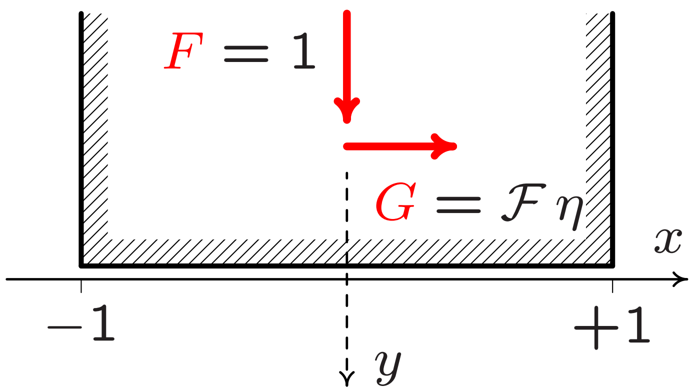
which defines a **scalar product** on $H^{-1/2}\big(]-1,1[\big)$ inducing a norm which is equivalent to the norm of $H^{-1/2}\big(]-1,1[\big)$.



The handling of (static) Coulomb friction

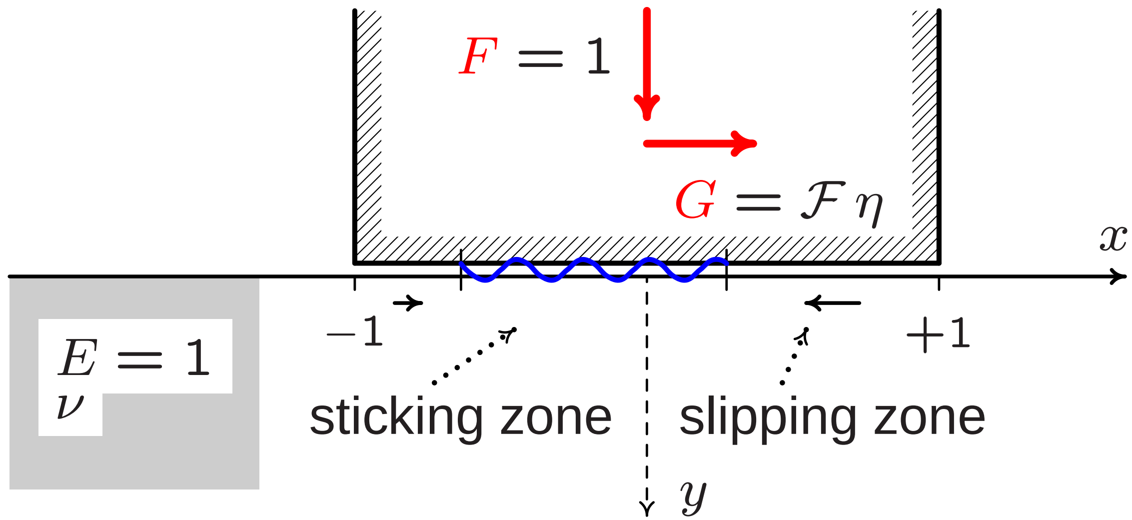
Find $t_x, t_y \in L^1(-1,1;\mathbb{R})$ and $\bar{u}_x, \bar{u}_y \in W^{1,1}(-1,1;\mathbb{R})$ satisfying for a.a. $x \in \,]-1,1[$:

- $$\frac{1}{\pi}\oint_{-1}^{1} \frac{t_x(t)}{t-x}\,\mathrm{d}t - \frac{1-2\nu}{2(1-\nu)}\,t_y(x) = \frac{1}{2(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_x(x),$$
  $$\frac{1}{\pi}\oint_{-1}^{1} \frac{t_y(t)}{t-x}\,\mathrm{d}t + \frac{1-2\nu}{2(1-\nu)}\,t_x(x) = \frac{1}{2(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_y(x),$$
- $$\int_{-1}^{1} t_y(t)\,\mathrm{d}t = F = 1, \qquad \int_{-1}^{1} t_x(t)\,\mathrm{d}t = G = \mathcal{F}\,\eta,$$
- $$\bar{u}_y(x) \geq 0, \qquad \bar{t}_y(x) \geq 0, \qquad \bar{u}_y(x)\,\bar{t}_y(x) \equiv 0,$$
- $$|\mathbf{t}_x(x)| \leq -\mathcal{F}\,t_y(x), \quad \text{and} \quad \left| \begin{array}{lcl} |\mathbf{t}_x(x)| < -\mathcal{F}\,t_y(x) & \Rightarrow & \mathbf{u}_x(x) = 0, \\ |\mathbf{t}_x(x)| = -\mathcal{F}\,t_y(x) & \Rightarrow & \mathbf{u}_x(x)\,\mathbf{t}_x(x) \leq 0. \end{array} \right.$$



$$\left(\eta \in \,]-1,1[\right)$$

General properties shared by any solution

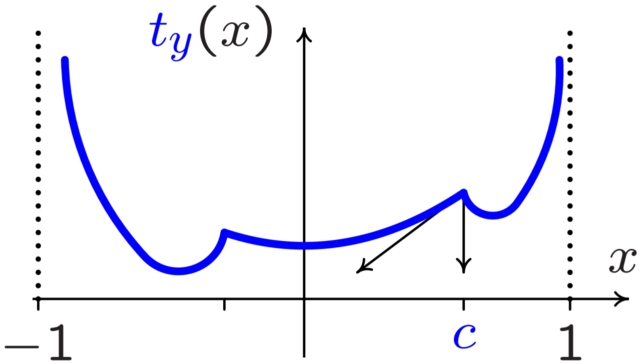


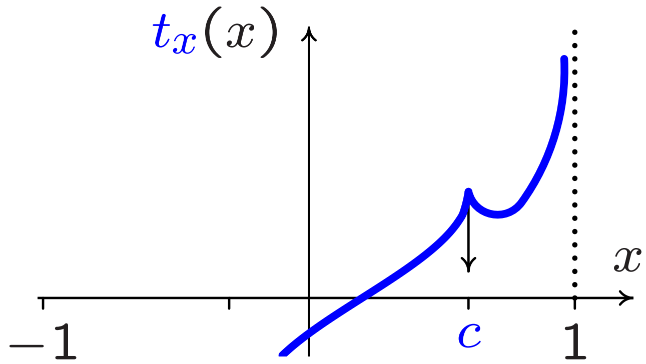
THEOREM

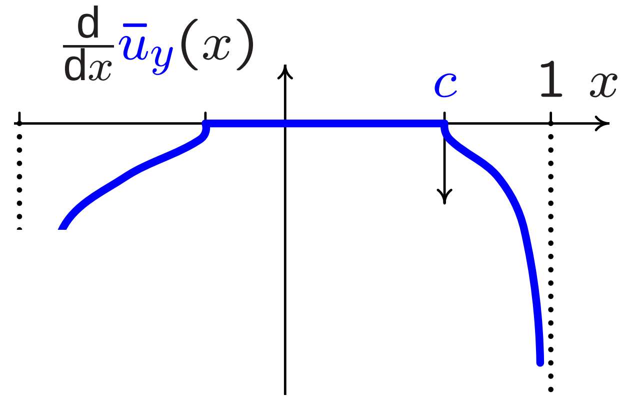
- Any solution of the problem **must** achieve active contact everywhere $\bar{u}_y \equiv 0$.
- Any solution of the problem **must** have a non-void connected sticking zone with nonempty interior.
- The sticking interval in any solution of the problem **must** be surrounded by two non-void peripheral **inwards slipping** zones that reach the two edges of the punch.

Regularity analysis of any solution

Set $\alpha = \frac{1}{\pi}\arctan\left(\mathcal{F}\,\frac{1-2\nu}{2(1-\nu)}\right) \qquad \in\,]0,1/2[\,.$







t_y has a left-deriv. at $x = c-$,

$t_y(x) \sim C(x-c)^{\frac{1}{2}-\alpha}$ as $x \to c+$

$t_y(x) \sim \frac{C}{(1-x)^{\frac{1}{2}-\alpha}}$ as $x \to 1-$

$t_x(x) \sim \frac{C}{(1-x)^{\frac{1}{2}-\alpha}}$ as $x \to 1-$

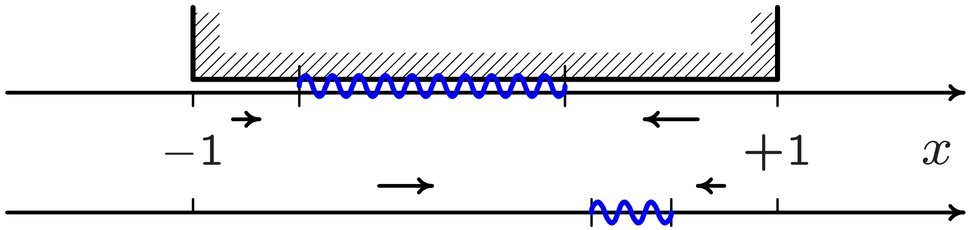
$\mathcal{F}t_y - t_x \sim C(c-x)^{\frac{1}{2}-\alpha}$ as $x \to c-$

$\frac{\mathrm{d}\bar{u}_y}{\mathrm{d}x} \sim \frac{C}{(1-x)^{\frac{1}{2}-\alpha}}$ as $x \to 1-$

$\frac{\mathrm{d}\bar{u}_y}{\mathrm{d}x} \sim C(x-c)^{\frac{1}{2}-\alpha}$ as $x \to c+$

In particular, t_x, t_y, $\frac{\mathrm{d}\bar{u}_y}{\mathrm{d}x}$ are locally $C^{\frac{1}{2}-\alpha} \subset H^{1/2}$ in $]-1,1[$ and belong to $L^2(-1,1;\mathbb{R})$ (**regularizing** effect of friction at corners).

Uniqueness analysis for this model problem



THEOREM There exists $\mathcal{F}_{\text{C}}(\nu) > 0$ such that, for all:

$$\mathcal{F} < \mathcal{F}_{\text{C}}(\nu),$$

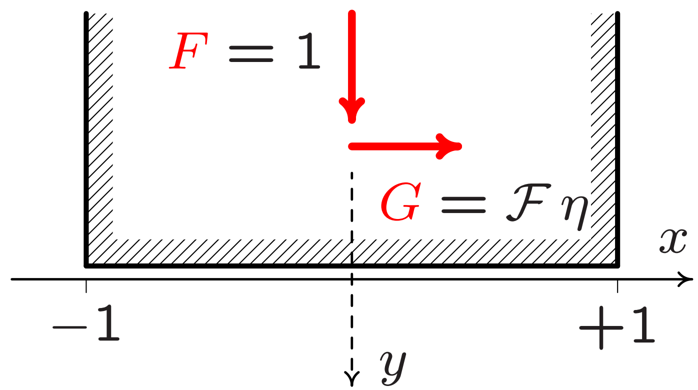
the sticking intervals of two distinct solutions (if any !) can not overlap.

$\Longrightarrow$ the number of solutions is at most countable.

A favourable limiting case

Find $t_x, t_y \in L^1(-1,1;\mathbb{R})$ and $\bar{u}_x, \bar{u}_y \in W^{1,1}(-1,1;\mathbb{R})$ satisfying for a.a. $x \in]-1,1[$:

- $$\frac{1}{\pi}\oint_{-1}^{1}\frac{t_x(t)}{t-x}\,\mathrm{d}t - \frac{1-2\nu}{2(1-\nu)}\,t_y(x) = \frac{1}{2(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_x(x),$$
  $$\frac{1}{\pi}\oint_{-1}^{1}\frac{t_y(t)}{t-x}\,\mathrm{d}t + \frac{1-2\nu}{2(1-\nu)}\,t_x(x) = \frac{1}{2(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_y(x),$$
- $$\int_{-1}^{1} t_y(t)\,\mathrm{d}t = F = 1, \qquad \int_{-1}^{1} t_x(t)\,\mathrm{d}t = G = \mathcal{F}\,\eta,$$
- $$\bar{u}_y(x) \geq 0, \qquad \bar{t}_y(x) \geq 0, \qquad \bar{u}_y(x)\,\bar{t}_y(x) \equiv 0,$$
- $$|\mathbf{t}_x(x)| \leq -\mathcal{F}\,t_y(x), \quad \text{and} \quad \left| \begin{array}{lll} |\mathbf{t}_x(x)| < -\mathcal{F}\,t_y(x) & \Rightarrow & \mathbf{u}_x(x) = 0, \\ |\mathbf{t}_x(x)| = -\mathcal{F}\,t_y(x) & \Rightarrow & \mathbf{u}_x(x)\,\mathbf{t}_x(x) \leq 0. \end{array} \right.$$

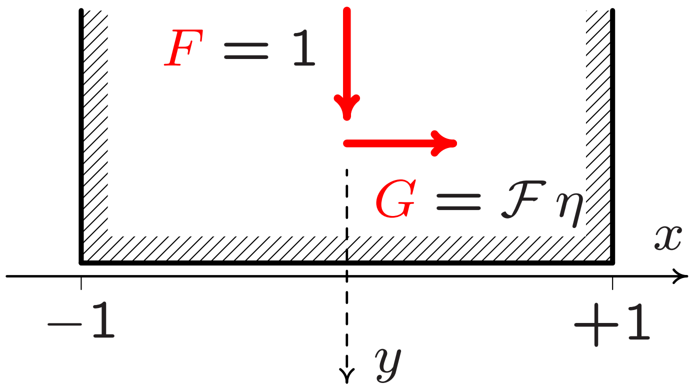


$\left(\eta \in]-1,1[\right)$

A favourable limiting case : incompressibility

Find $t_x, t_y \in L^1(-1,1;\mathbb{R})$ and $\bar{u}_x, \bar{u}_y \in W^{1,1}(-1,1;\mathbb{R})$ satisfying for a.a. $x \in]-1,1[$:

- $$\frac{1}{\pi}\oint_{-1}^{1} \frac{t_x(t)}{t-x}\,\mathrm{d}t = \frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_x(x),$$
  $$\frac{1}{\pi}\oint_{-1}^{1} \frac{t_y(t)}{t-x}\,\mathrm{d}t = \frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}x}\bar{u}_y(x),$$
- $\int_{-1}^{1} t_y(t)\,\mathrm{d}t = F = 1, \qquad \int_{-1}^{1} t_x(t)\,\mathrm{d}t = G = \mathcal{F}\,\eta,$
- $\bar{u}_y(x) \geq 0, \qquad \bar{t}_y(x) \geq 0, \qquad \bar{u}_y(x)\,\bar{t}_y(x) \equiv 0,$
- $|\mathbf{t}_x(x)| \leq -\mathcal{F}\,t_y(x)$, and
  $$\left| \begin{array}{lll} |\mathbf{t}_x(x)| < -\mathcal{F}\,t_y(x) & \Rightarrow & \mathbf{u}_x(x) = 0, \\ |\mathbf{t}_x(x)| = -\mathcal{F}\,t_y(x) & \Rightarrow & \mathbf{u}_x(x)\,\mathbf{t}_x(x) \leq 0. \end{array} \right.$$

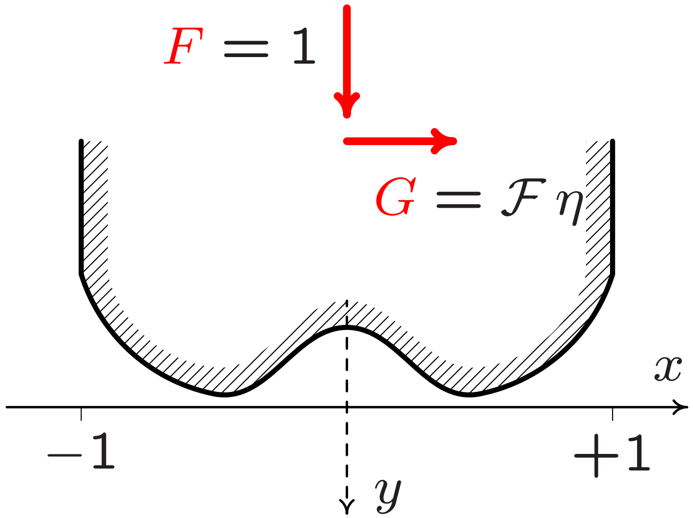


$\left(\eta \in]-1,1[\right)$

A favourable limiting case : incompressibility

THEOREM The frictional contact problem for the **incompressible** isotropic elastic 2D half-space and obstacle of arbitrary shape $\phi \in H^{1/2}$, admits a unique solution for all:

$$\mathcal{F} > 0.$$



$$\Big(\eta \in \,]-1, 1[\Big)$$