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Unilateral Problems in Structural Analysis
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Singular stress fields in masonry vaults

M. Angelillo, A.Fortunato, M.Lippiello

Dept Civil Eng, Università di Salerno

Basic references

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NENT model

Masonry material is brittle and characterized by a very small and aleatory value of toughness; cracks are kind of physiological in masonry, and are likely to open up in the material under the effect of the working loads solely.
As a first approximation to this behavior the no–tension material has been proposed. This crude model that describes the material as elastic in compression but incapable of sustaining tensile stresses, was first rationally introduced by Heyman in [1] and studied by Di Pasquale in [2].

The idea of a no–tension (NT) material underlies more or less consciously the design of masonry structures since antiquity, particularly for vaulted masonry structures and arches.
Based on the NT model the safety of the structure is a problem of geometry rather than of strength of materials, in keeping with the spirit of the "rules of proportions" used by ancient architects for masonry design.

Essentially the constitutive restrictions defining NT materials are as follows:
- The stress tensor is assumed to be negative semidefinite.
- The total strain is decomposed additively into the sum of its elastic and anelastic parts, the elastic part depending linearly upon the stress.
- A normality law to the elastic stress domain is imposed on the anelastic strain that turns to be positive semidefinite.

NT model in 2d (walls)

The basic restriction for NT materials is that the stress **T** is negative semidefinite:

$$\mathbf{T} \in NSym$$

Other asssumptions.The infinitesimal strain is decomposed additively into two parts:

$$\mathbf{e} = \underbrace{\boldsymbol{\varepsilon}}_{\text{elastic part}} + \underbrace{\boldsymbol{\lambda}}_{\text{fracture part}}$$

The elastic part is linerly related to stress

$$\boldsymbol{\varepsilon} = \mathbb{A}[\mathbf{T}]$$

The total anelastic strain (fracture) satisfies a normality rule …equivalent to:

$$\boldsymbol{\lambda} \in PSym$$

$$\mathbf{T} \cdot \boldsymbol{\lambda} = 0$$

NT materials & Limit Analysis in 2d

As shown by Del Piero in [3], the Static and Kinematic theorems of Limit Analysis hold, with some peculiarities, for bodies composed of NT material. In particular the safety of the applied load can be tested through the first theorem.

Notice that the strenght of a NT material is infinite, therefore infinite stresses are admissible, as long as they are compressive, and singular stress fields are perfectly adequate for testing the Static theorem.
Such singular stress fields are not only admissible but also acceptable as equilibrium solutions if the material is considered as rigid in compression.
The use of singular stress fields for the problem of equilibrium of NT materials has been recently proposed by Lucchesi et al [6]. Here we propose a sort of graphycal way to construct them, based on the Airy stress formulation.

Singular stress fields in 2d

Singular stress fields are bounded measures, that is

$$\underbrace{|\mathbf{T}|}_{\text{TOTAL VARIATION}}(\Omega) < \infty$$

The set of bounded (Radon) measures is called $\mathcal{M}_b(\Omega)$. If **T** belongs to $\mathcal{M}_b(\Omega)$ then **T** consists of two parts:

$$\mathbf{T} = \underbrace{\mathbf{T}^a}_{\text{ABSOLUTELY CONTINUOS}} + \underbrace{\mathbf{T}^s}_{\text{SINGULAR}}$$

We shall restrict to the simpler case in which the support of the singular part is a 1d set. In particular we consider singular stress fields that are line Dirac Deltas over a finite number of rectifiable curves.

In equilibrium. the stress **T** must be balanced. In absence of body forces:

$$\text{div}\,\mathbf{T} = \mathbf{0} \;\ldots.$$

…extend the divergence free condition to singular stress fields (Lucchesi et al)

Singular stress fields and ST methods in 2d

Why do we like such fields ?

Because they are easy to be obtained and we can interpret them as contact forces acting on 1D structures arising inside the body Ω … in the same spirit of **S**trut and **T**ies methods (See [4]).
Except that here the structures can only be composed by truss elements.
The axial forces inside such structures can be made to balance the external loads and are equivalent to equilibrated stress fields becoming singular along the structure axis.

The rationale for using singular stress fields for elastic problems is to consider them as approximations of the true stress, and that there is a way in which these "infininite" stresses can be smeared over some influence area.
A systematic and convergent way to do so is offered, in Classical LE, by the LSM (Angelillo et al [5]) …. relaxed form of Complementary Energy

Singular stress fields in 2d. Airy stress function

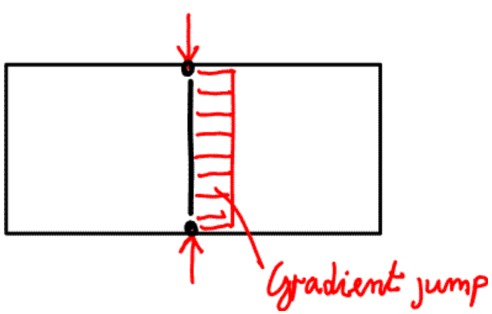
A simple way to obtain balanced (divergence free) singular stresses and visualize them is to use the Airy's stress function formulation (assume **b**=0)

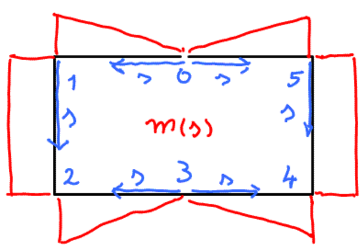
$$T_{11} = F_{,22}, \quad T_{22} = F_{,11}, \quad T_{12} = -F_{,12}$$

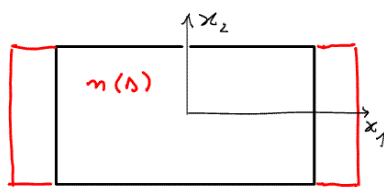
$$F\Big|_{\partial\Omega} = m, \quad \frac{dF}{dn}\Big|_{\partial\Omega} = -n$$

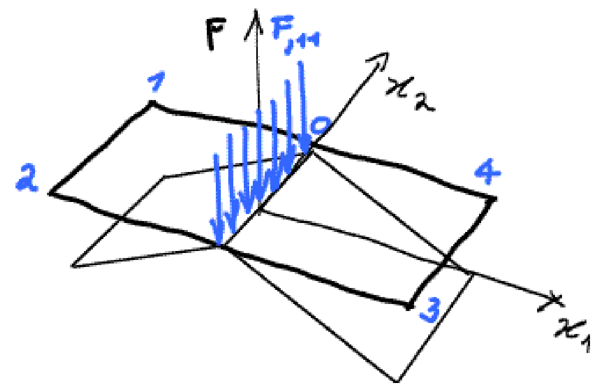
… and consider F $\in$ SBH(Ω), that is F $\in$ C°(Ω) and the Hessian of F , H(F) $\in$ $\mathcal{M}_b(\Omega)$.

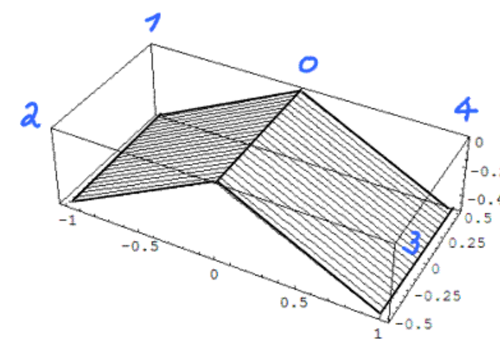
Example. Rectangular strip under equal and opposite concentrated forces





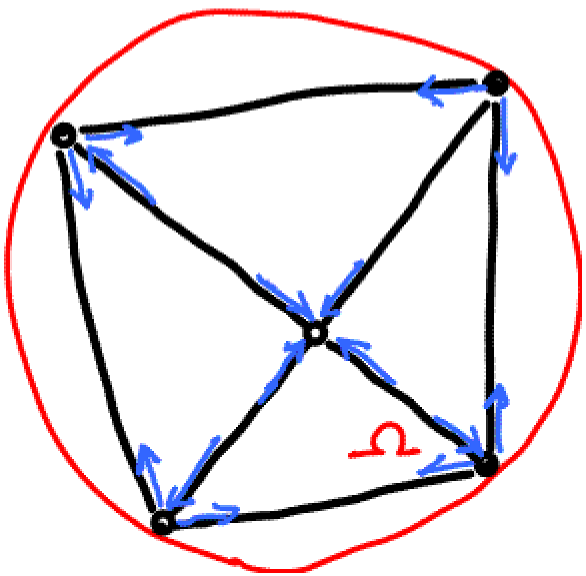






Singular stress fields: simple examples

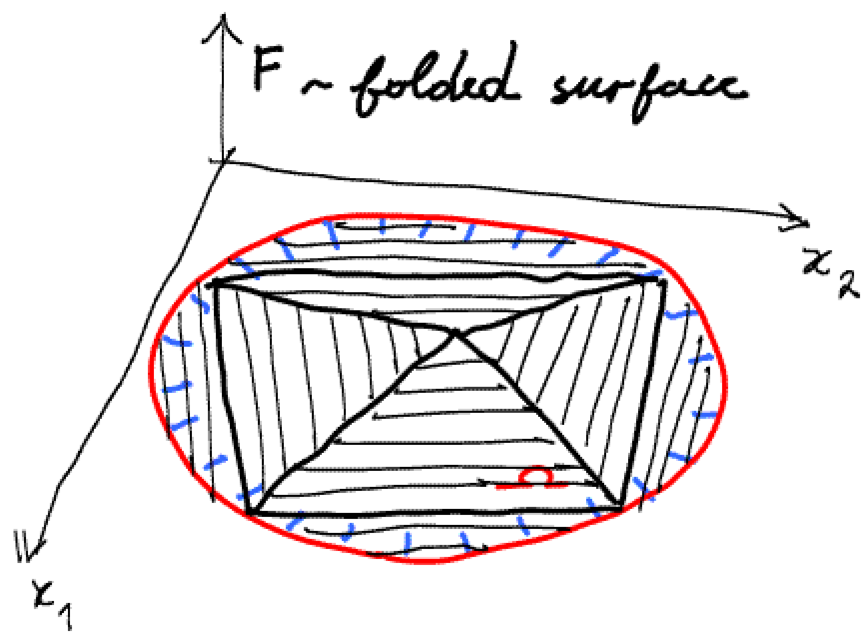
Self stress in an unloaded body **1**



Airy’s stress function formulation

$$T_{11} = F_{,22}\,, \quad T_{22} = F_{,11}\,, \quad T_{12} = -F_{,12}$$

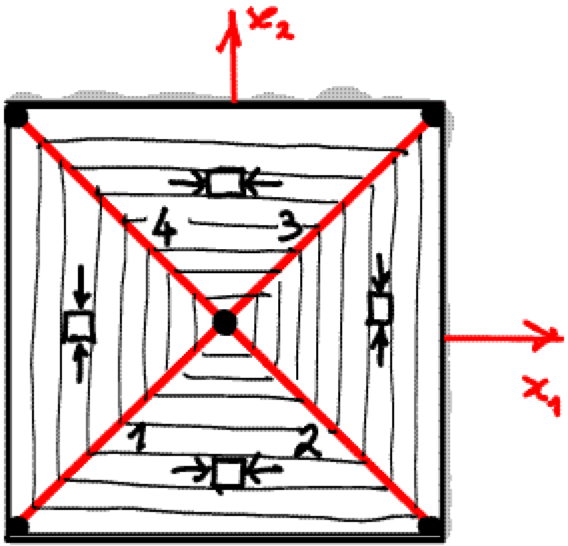
$$F\Big|_{\partial\Omega} = m\,, \quad \frac{dF}{dn}\Big|_{\partial\Omega} = -n$$

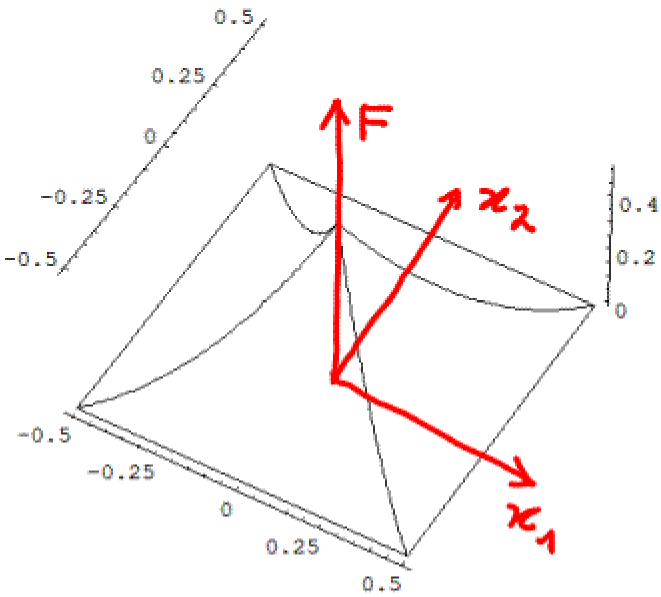


Singular stress fields: simple examples

Self stress in an unloaded body **2**

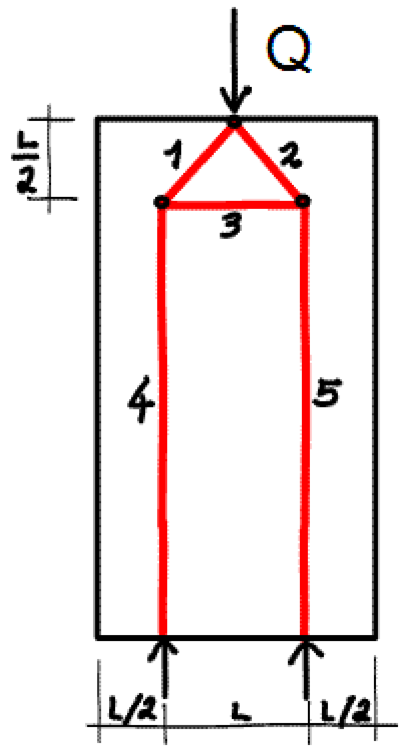
Combination of "singular" and "diffuse"(absolutely continuous) stress



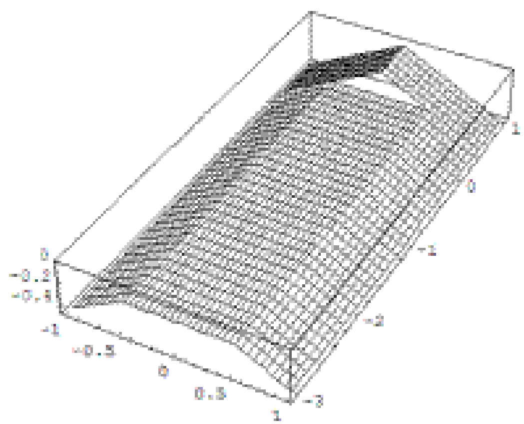


Differences between Classical LE and NT models

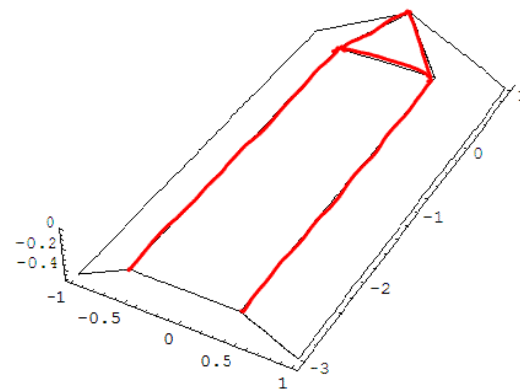
Diffusion of concentrated load in a strip **(CLE)**



Boundary data **m** and **n**



Insert a compensating triangle



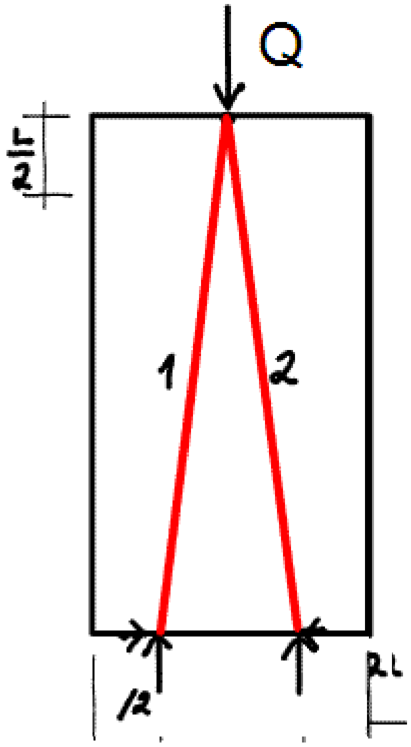
Prolongate the data with a **ruled** surface

Differences between Classical LE and NT models

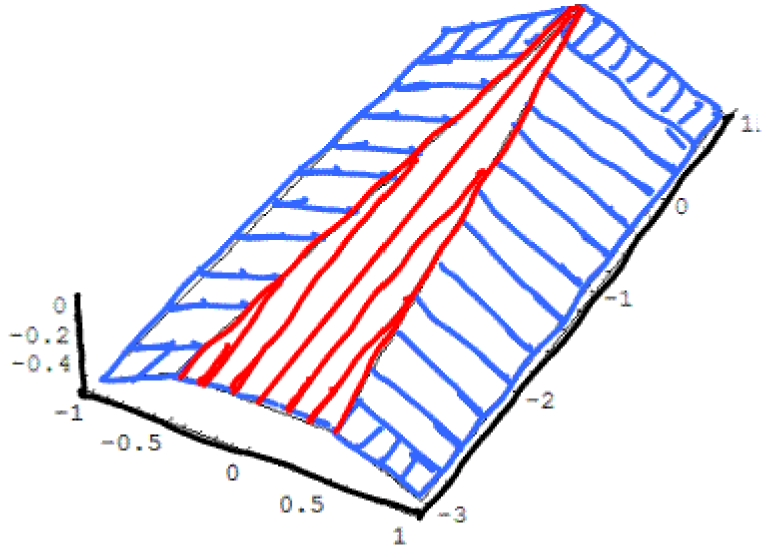
Diffusion of concentrated load in a strip **(NT material)**

Due to NT constraint Airy's stress function F must be concave

According to the Static Theorem if a Concave F can be found, in equilibrium with the given loads, then the load is safe



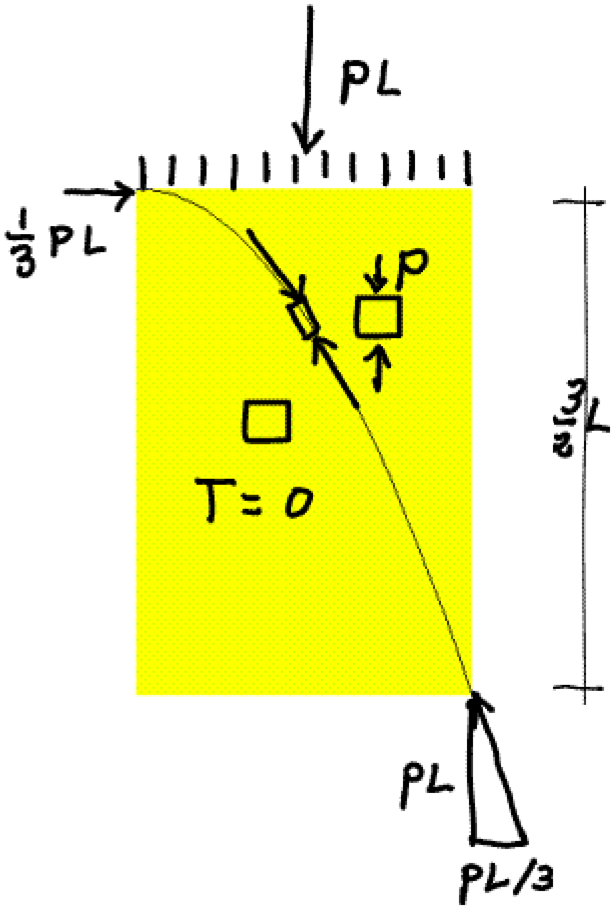
The "structure is created by the intersection of the three ruled surfaces (planes)

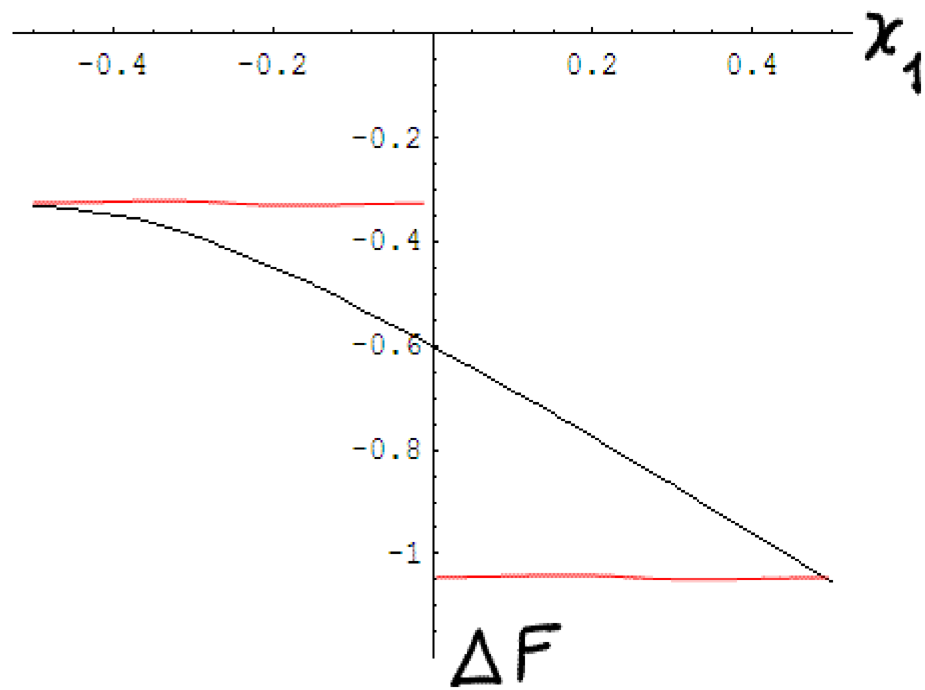


There is no Saint Venant effect

A basic example:
The rocking of a NT wall due to a horizontal load

The wall is loaded by a uniform vertical load at the top base …
and by a horizontal concentrated force at the left corner

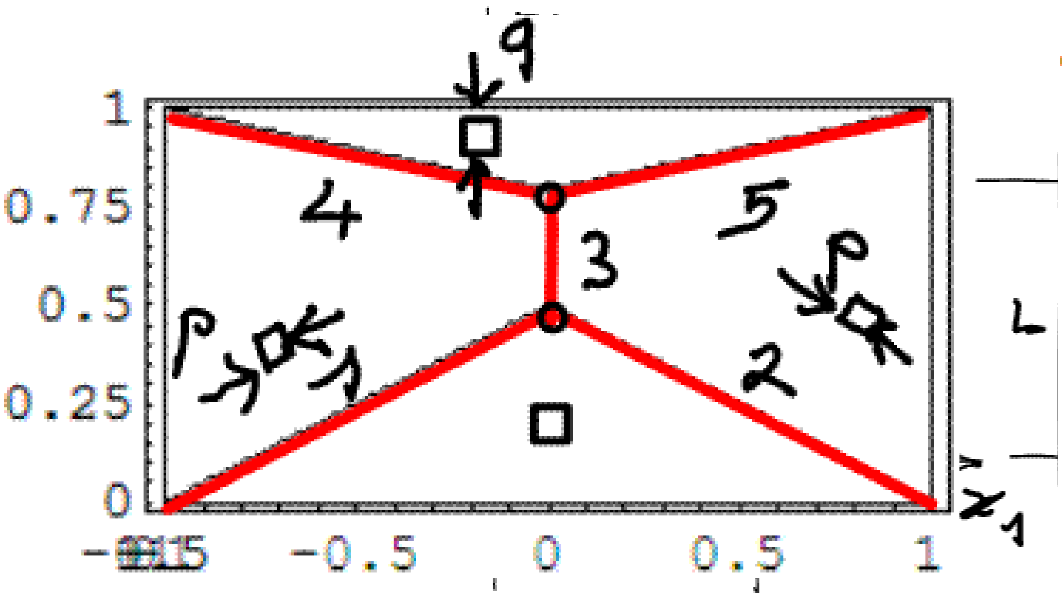


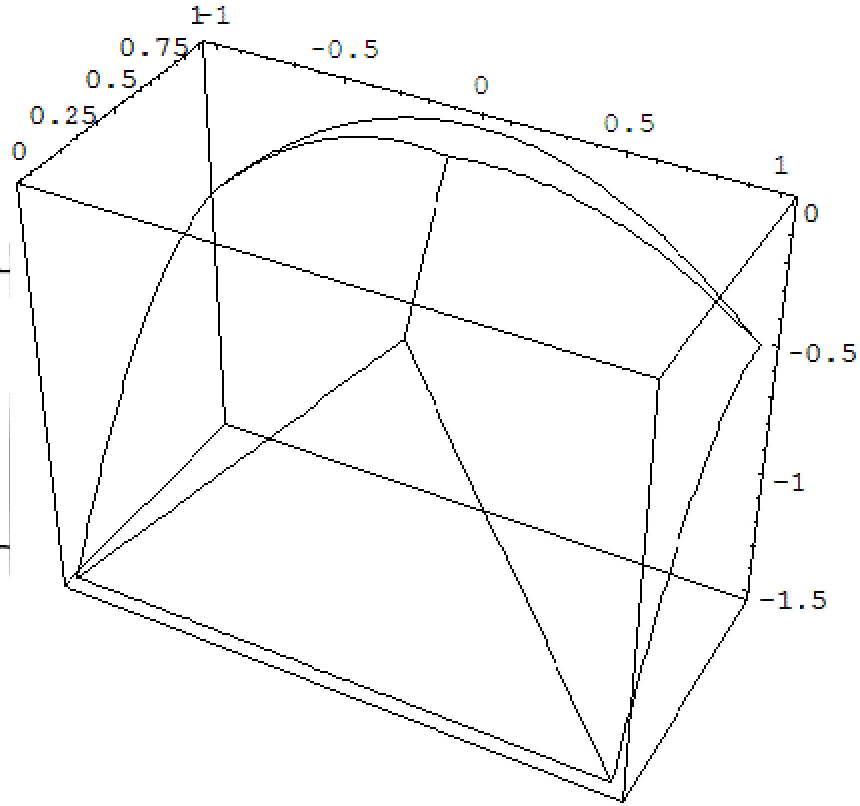


The "structure" is created by the intersection of the two ruled surfaces

Another example:
NT wall beam subject to a transverse load

The wall is loaded by a uniform vertical load at the top base …
and by uniform horizontal loads at the two ends. Uniform vertical loads are also given at the two ends





Prolongating data with ruled surfaces …

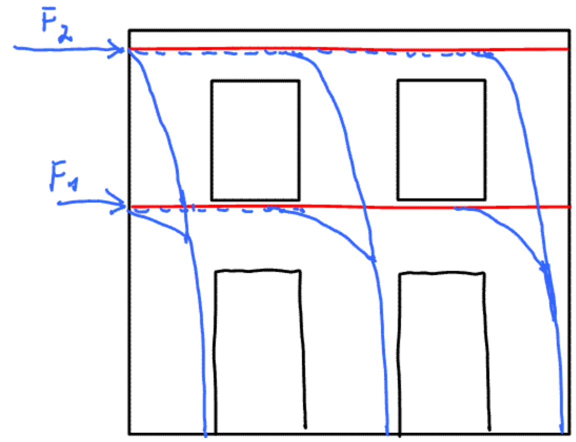
support of gradient discontinuities gives "structure"

Walls

Allowing for stress functions F's that can have folds (C° (SBH) functions), singular stress fields that are statically admissible (equilibrated and compressive) are easily created.

We may think of the load as being carried by a sort of truss structure, coincident with the support line of the jump set of gradF.
The value of the jump across the line measures the axial force inside the structure…arching.

In more complex cases, such as walls with openings, the generation of the stress surface can be automated. Often the static multiplier gets close to the kinematic one (obtained by block rigid body rotations).



Vaults

So as a masonry arch can be idealized as an inverted chain (the thrust line) a masonry vault can be modeled as a membrane *S*. If the vault is unilateral the membrane must be compressed at any point and contained between the estrados and intrados surfaces of the vault.

The main difference between a chain and a membrane is that the chain is underdetermined and the equilibrium under a given load is assured only if the chain takes the "equilibrium shape".
On the other hand a membrane *S* can equilibrate a wide range of loads over a specified shape. The membrane *S* becomes undertetermined if is unilateral:

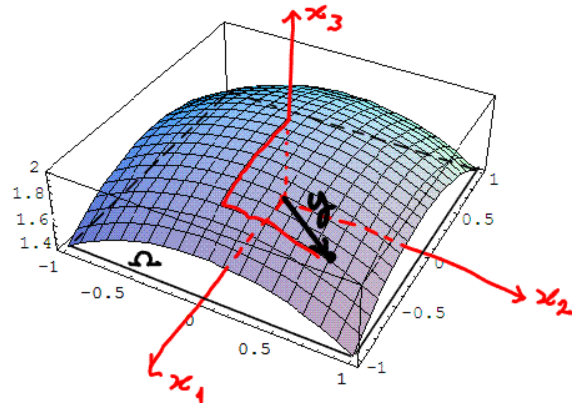
"the shape *S* can be given as long as *S* is under compression for the given load, but it must adapt to loads in regions over which the stress becomes uniaxial".

Vaults

Geometry.

…Then it is useful to adopt a description of *S* for which the "planform" Ω is fixed and the rise of the membrane can become a further variable:

$$\mathbf{y} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + f(x_1,x_2)\,\mathbf{e}_3 \quad , \quad (x_1,x_2) \in \Omega \quad .$$



The couple of parameters $\{\theta^1 = x_1,\ \theta^2 = x_2\}$ define a curvilinear system over *S* whose associated natural and reciprocal bases are

$$a_1 = e_1 + f_{,1}\, e_3 \quad , \quad a_2 = e_2 + f_{,2}\, e_3 \quad ,$$

$$a^1 = \frac{1}{J^2}\left((1+f_{,2}^2)\, e_1 - f_{,1} f_{,2}\, e_2 + f_{,1}\, e_3\right), \quad a^2 = \frac{1}{J^2}\left(-f_{,1} f_{,2}\, e_1 + (1+f_{,1}^2)\, e_2 + f_{,2}\, e_3\right),$$

$$J = |a_1 \times a_2| = \sqrt{1 + f_{,1}^2 + f_{,2}^2} \quad , \quad a_3 = \frac{a_1 \times a_2}{J} \quad .$$

Vaults

Forces

We consider the equilibrium of *S* under the action of surface forces per unit planform area

$$\mathbf{b} = b_{(1)}\, \mathbf{e}_1 + b_{(2)}\, \mathbf{e}_2 + b_{(3)}\, \mathbf{e}_3 \quad .$$

The generalized membrane stress is denoted

$$\mathbf{T} = T^{\alpha\beta}\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta \quad .$$

Projecting the vector equilibrium equation

$$\partial/\partial\theta^\gamma \left(T^{\alpha\beta}\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta\right) \mathbf{a}^\gamma + \mathbf{b}/J = \mathbf{0} \quad ,$$

over the three no-coplanar directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{a}_3$, get

$$\left(J T^{\alpha\beta}\right)_{,\beta} + b_{(\alpha)} = 0 \quad ,$$

$$J T^{\alpha\beta} f_{,\alpha\beta} - f_{,\alpha}\, b_{(\alpha)} + b_{(3)} = 0 \quad .$$

Vaults

Forces

On introducing the pseudo-stresses

$$S^{\alpha\beta} = J\,T^{\alpha\beta} \;,$$

can rewrite the equilibrium equations as

$$S^{\alpha\beta}{}_{,\beta} + b_{(\alpha)} = 0 \;,$$
$$S^{\alpha\beta} f_{,\alpha\beta} - f_{,\alpha} b_{(\alpha)} + b_{(3)} = 0 \;,$$

that is essentially Pucher's form of membrane equilibrium.

The first two equations have the same form of the plane case, then if **b** is a potential load they can be solved in terms of a stress function F. In the simple case of vertical loads ($b_{(1)}=b_{(2)}=0$):

$$S^{11} = F_{,22} \;,\quad S^{22} = F_{,11} \;,\quad S^{12} = -F_{,12} \;.$$

Substituting the Airy's solution into the transverse equilibrium equation (with $p=-b_{(3)}$)…

$$F_{,22} f_{,11} + F_{,11} f_{,22} - 2 F_{,12} f_{,12} = p \;,$$

Vaults

Forces

… this is a second order pde both in F and f, symmetric by interchanging F with f. If the shape of *S* is given, that is f is given, a restricted class of F's is obtained by solving it.

If the stress corresponding to such a restricted class of stress functions F is compressive, the equilibrium solution is acceptable for the unilateral membrane. If not the shape must be changed … at least in the regions where the stress is not negative semidefinite.

Another possible approach to obtain an admissible stress field is to start from a statically admissible pseudo-stress (through the assignement of the corresponding stress function F) and determine, by solving the pde of transverse equilibrium, a restricted class of shapes f that can fit inside the vault.

To do so we need to translate the unilateral restrictions on **T** onto constraints on the pseudo-stress **S**.

Vaults

Unilateral restrictions

The unilateral assumptions consist for the membrane *S* into the following restrictions on **T**:

$$\mathrm{tr}\,\mathbf{T} \le 0 \quad , \quad \det \mathbf{T} \ge 0 \ .$$

In terms of the contravariant components of **T** they reduce to the form:

$$g_{\alpha\beta}\, T^{\alpha\beta} \le 0 \quad , \quad J^2\,(T^{11}T^{22} - (T^{12})^2) \ge 0 \quad ,$$

that, in terms of S and in more esplicit form, become

$$\bullet\nu(\mathbf{S}) = (1+f_{,1}{}^2)\, S^{11} + (1+f_{,2}{}^2)\, S^{22} + 2\, f_{,1} f_{,2}\, S^{12} \le 0 \quad ,$$

$$\mu(\mathbf{S}) = S^{11}\, S^{22} - (S^{12})^2 \ge 0 \ .$$

The equation $\mu(\mathbf{S})=0$ defines the boundary of a right circular cone $\mathcal{C}$ of axis $S^{11}=S^{22}$ and “opening angle” $\pi/2$. The inequality $\mu(\mathbf{S}) \ge 0$ restricts **S** to belong to the interior or to the boundary of $\mathcal{C}$.

Vaults

Unilateral restrictions

Noticing that $\nu(\mathbf{S}) = 0$ can be written as:

$$\mathbf{S} \cdot (\mathbf{I} + \mathbf{M}) = 0 \quad ,$$

where $\mathbf{M} = \mathbf{m} \otimes \mathbf{m}$, with $\mathbf{m} = f_{,1}\, \mathbf{e}_1 + f_{,2}\, \mathbf{e}_2$, $\nu(\mathbf{S}) = 0$ is the equation of the plane Π orthogonal to $(\mathbf{I} + \mathbf{M})$ and passing through the origin O. Since $(\mathbf{I} + \mathbf{M})$ is positive definite Π intersects the cone $\mathcal{C}$ only at O and then the condition $\nu(\mathbf{S}) \leq 0$ can be substituted by the simpler inequality

$$\nu{*}(\mathbf{S}) = S^{11} + S^{22} \leq 0 \quad .$$

Therefore the restrictions on **S** assume the same form of the plane case and the Airy's stress function must be concave:

$$F_{,11} + F_{,22} \leq 0 \quad \& \quad F_{,11} F_{,22} - F_{,12}^2 \geq 0 \quad .$$

Based on the strict analogy with the plane case the planform Ω can be split, as in the 2d case, into the three disjoint regions Ω_1, Ω_2, Ω_3 … and the isostatic lines of compression, in the non degenerate regions Ω_2 , are straight if $b_{(1)} = b_{(2)} = 0$.

Vaults

A trivial (but fundamental) example: parabolic shape.
There is a particular shape of the vault for which a very simple membrane equilibrium solution under uniform vertical load (p.u. planform area) exists: is the domical vault (*"volta a vela"*).

In this case the form of the shape f and stress F can be taken the same:

$$F = \sigma/2\ (L^2-x_1^2-x_2^2)\ ,\ f = H/L^2\ (L^2-x_1^2-x_2^2)\ ,$$

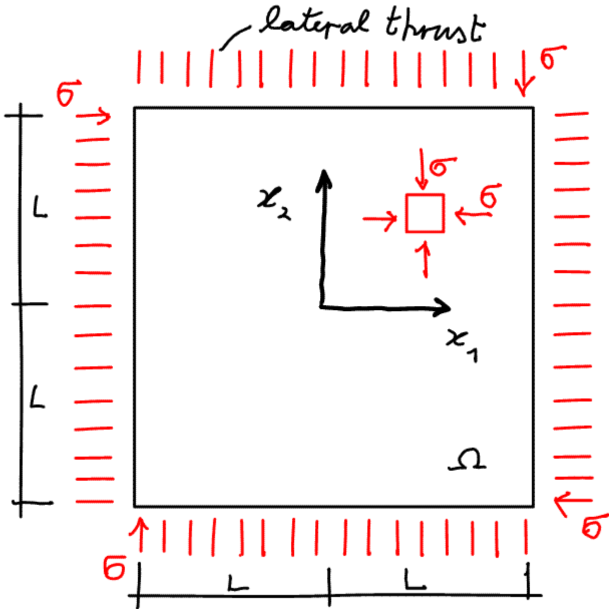
where, for equilibrium with the uniform load p°:

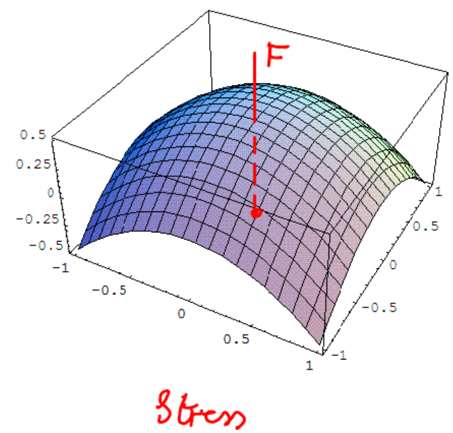
$$\sigma = p°\ L^2/(4\ H)\ .$$

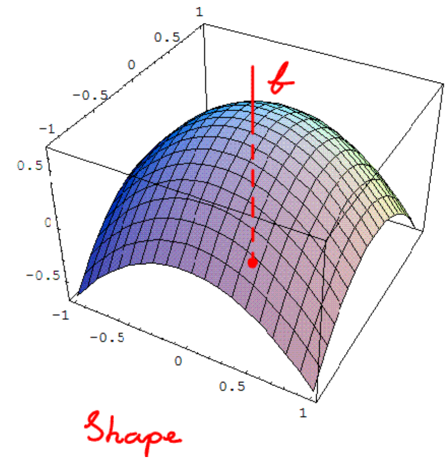
$$F_{,22}\, f_{,11} + F_{,11}\, f_{,22} - 2\, F_{,12}\, f_{,12} = p\ ,$$

"Volte a vela". Royal Palace Napoli

Vaults

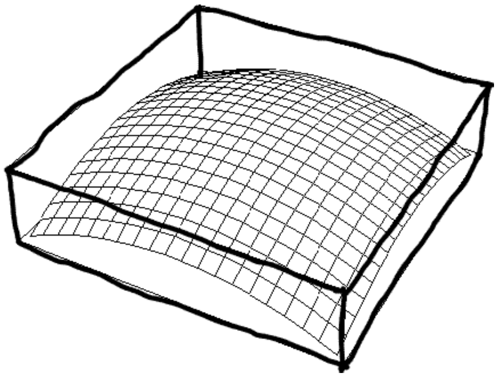






This simple equilibrium solution can be adapted to different shapes of the vault as long as the parabolic surface *S* can be fitted in between the extrados and intrados surfaces.

… thickness large enough …
… a parabolic shape can always be fitted inside a plate …. large lateral forces for H small



Vaults

Singular stress.

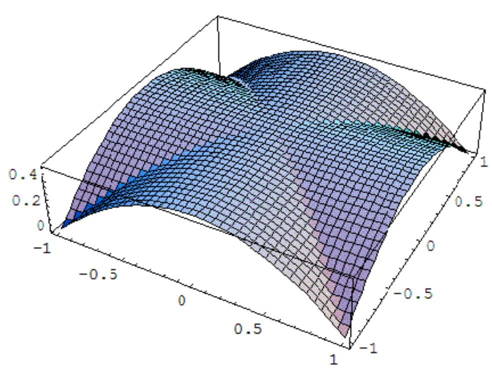
The main difference between the equilibrium problem for unilateral membranes and for unilateral plane bodies is that, in the first case, the basic equilibrium equations are supplemented by the transverse equilibrium equation … for vertical loads, in terms of the Airy's solution:

$$F_{,22}\, f_{,11} + F_{,11}\, f_{,22} - 2\, F_{,12}\, f_{,12} = p \quad ,$$

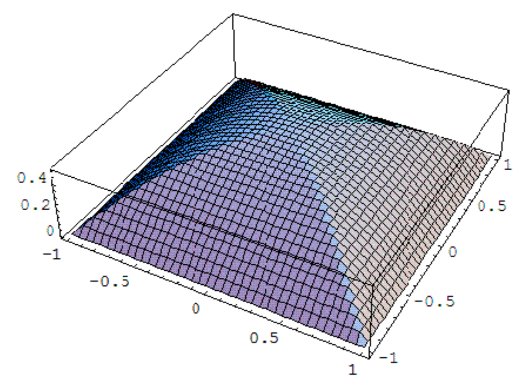
The question arise if for unilateral membranes singular stress fields are acceptable as s.a. stress fields as in the plane case.

Notice that not only the Airy's surface but also the surface S can have folds …singular Hessian. Most immediate examples:

cross vault



cloister vault



Vaults

Singular stress.

If the vault is thick enough a smooth surface *S* could always be fitted inside it, but a folded *S* may be necessary.

Actually we would like to start by giving the stress (that is F) and find f that satisfy transverse equilibrium.

To fix ideas assume that $F_{,11}$ is a line Dirac delta along x_2. In order to satisfy

$$F_{,22} f_{,11} + F_{,11} f_{,22} - 2 F_{,12} f_{,12} = p \quad ,$$

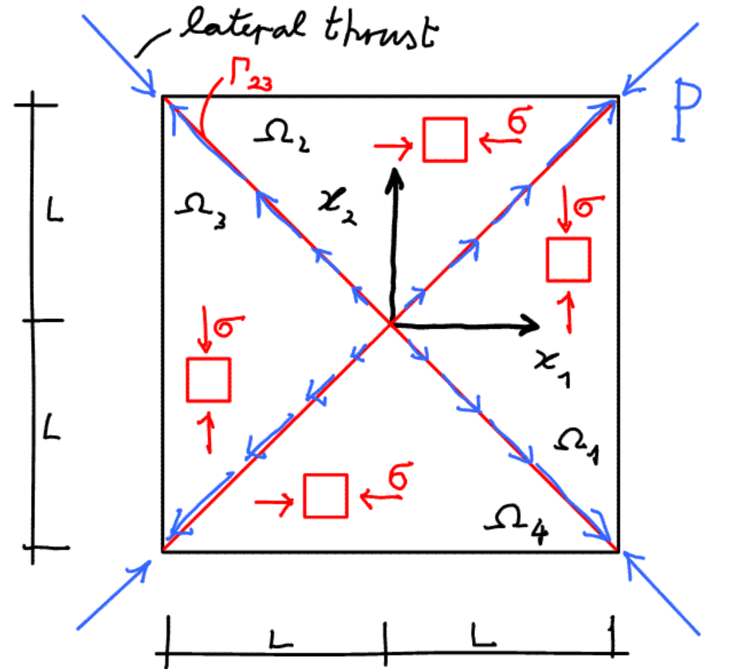
either

- $f_{,22}=0$,
- p singular along x_2 and canceling $F_{,11}$ $f_{,22}$,
- f,11 singular and $F_{,22}$ $f_{,11}$ canceling $F_{,11}$ $f_{,22}$.

Vaults

Cross vault.

For a cross vault (*"volta a crociera"*) a sensible choise for the pseudo-stress could be the one depicted in the figure….



..to which there corresponds the stress function

$$F = \begin{cases} \sigma/2\,(L^2 - x_1^2) & , x_2 < x_1 \;\&\; x_2 > -x_1 \;, \\ \sigma/2\,(L^2 - x_2^2) & , \text{otherwise} \;. \end{cases}$$

The singular part of the Hessian of F on the interface Γ_{23} reads

$-\delta(\Gamma_{23})$ ⌦ $2\,\sigma\, x_1\, \mathbf{e}_{1'} \otimes \mathbf{e}_{1'}$

Vaults

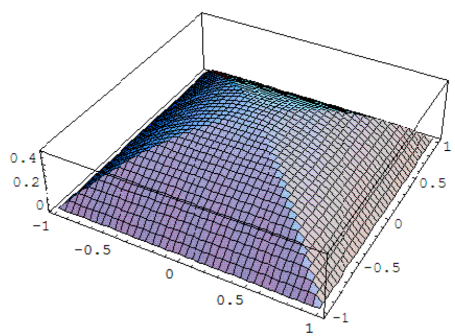
Cross vault.

In this case the form of the shape f is

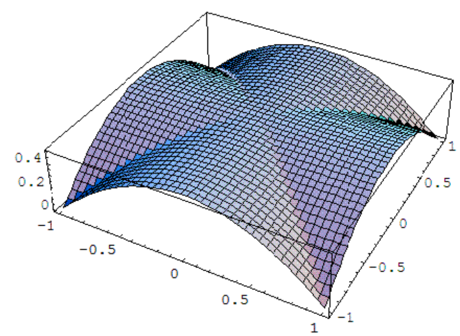
$$f = \begin{cases} H/L^2\,(L^2 - x_2^2) & , \mathbf{x} \in \Omega_1 \cup \Omega_3 \\ H/L^2\,(L^2 - x_1^2) & , \mathbf{x} \in \Omega_2 \cup \Omega_4 \end{cases}$$

… the two singular parts cancel each other on each interface:

- $\delta(\Gamma)$ ⌦ $2\,\sigma\,x\,(-H/L^2) + \delta(\Gamma)$ ⌦ $2\,H/L^2\,x\,(-\sigma) == 0$ …



F (stress)



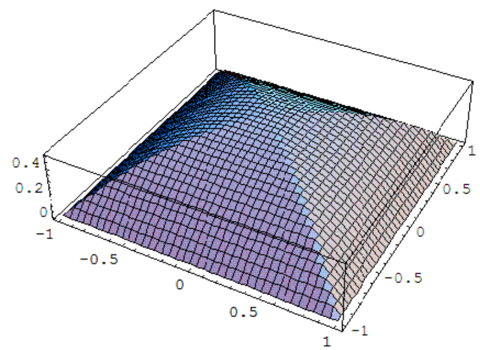
f (shape)

… while, for equilibrium with the uniform load p°: $\sigma = p°\,L^2/(2\,H)$.

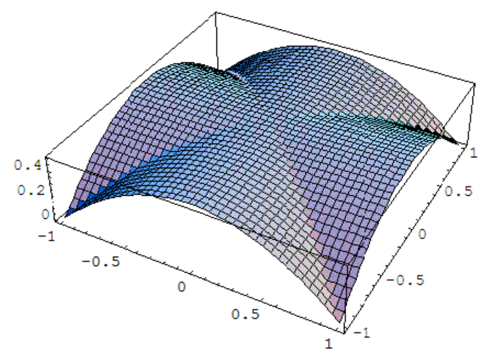
Vaults

Cloister vault.

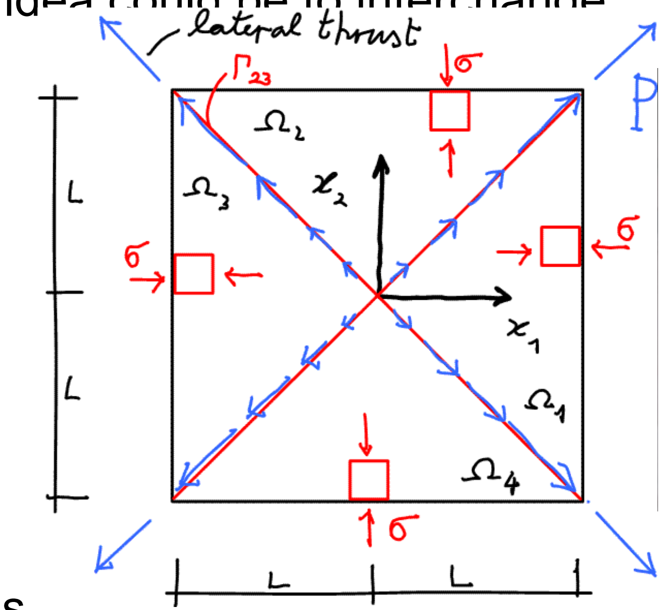
For a cloister vault (*"volta a padiglione"*) the first idea could be to interchange stress and shape as shown in the figure….



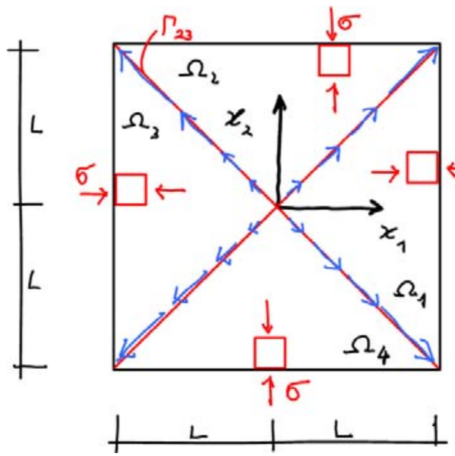
f (shape)

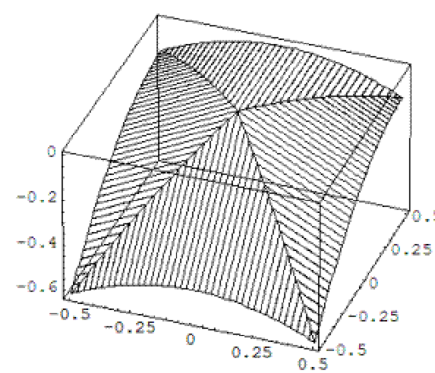


F (stress) ?

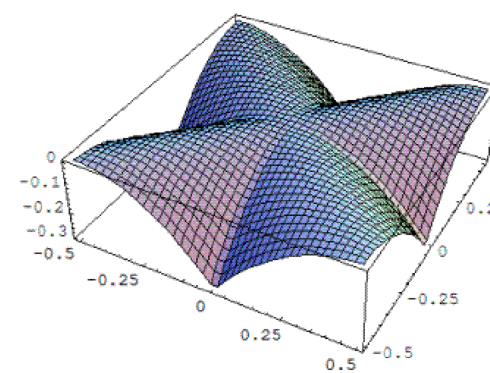


.. it does not work .. F is NON-CONCAVE .. stress …





F (stress)



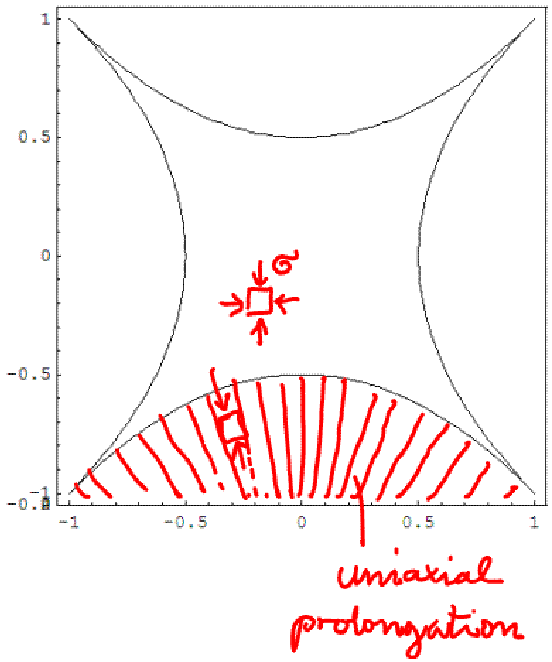
f (shape)

Vaults

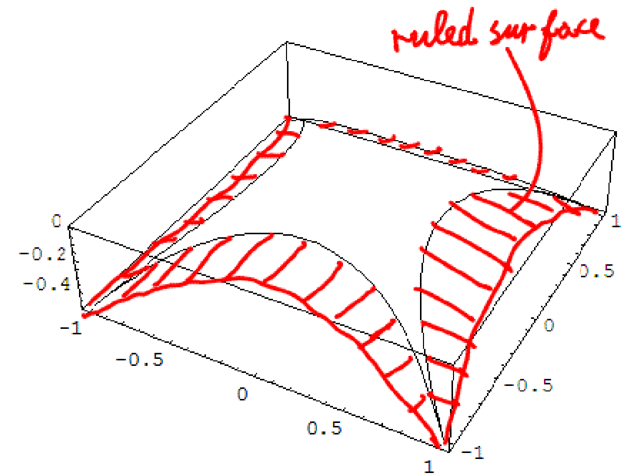
Cloister vault.

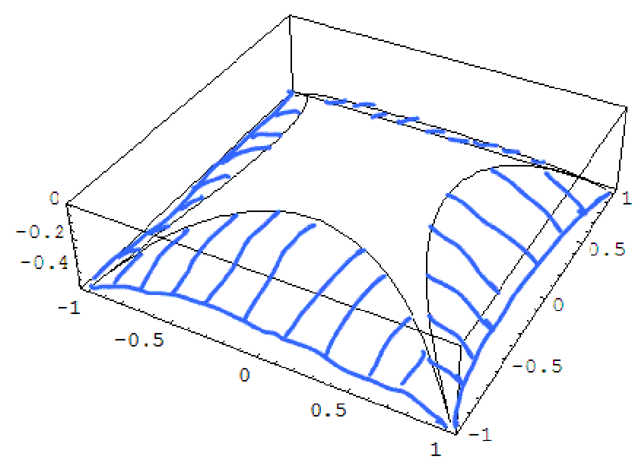
Another way: use parabolic equilibrium solution in a central part. Extend to boundary with uniaxial prolongation …

Corresponding stress function F…



Equilibrium shape …. $f(x_1,x_2)$ ….





Cylindrical description (axial symmetry)

Cylindrical coordinates $\{\theta^1,\theta^2\}=\{\rho,\theta\}$,

$$y_1 = \rho\cos\vartheta\,,\qquad y_2 = \rho\,\mathrm{sen}\,\vartheta\,,\qquad y_3 = f(\rho,\vartheta)\,,$$

$$\sigma_{11} = \frac{1}{\rho}F_{,1} + \frac{1}{\rho^2}F_{,22}\,,\qquad \sigma_{22} = F_{,11}\,,\qquad \sigma_{12} = -\left(\frac{1}{\rho}F_{,2}\right)_{,1}\,.$$

Physical stress components in terms of Airy's stress function

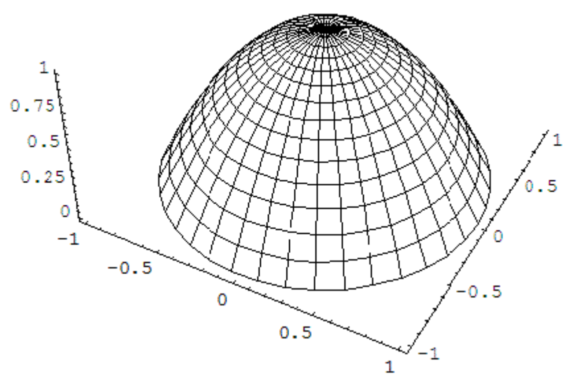
Transverse equilibrium:

$$\left(\frac{1}{\rho}F_{,1} + \frac{1}{\rho^2}F_{,22}\right)f_{,11} + F_{,11}\left(\frac{1}{\rho}f_{,1} + \frac{1}{\rho^2}f_{,22}\right) - 2\left(\frac{1}{\rho}F_{,2}\right)_{,1}\left(\frac{1}{\rho}f_{,2}\right)_{,1} - p = 0\,.$$

Parabolic dome.

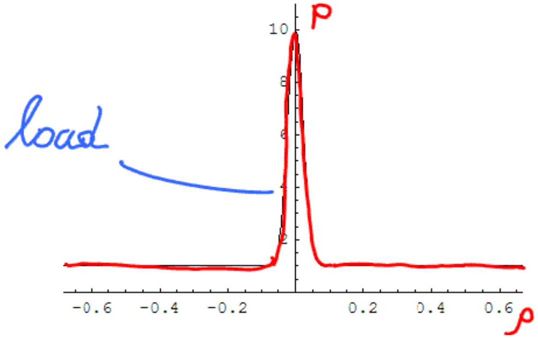
$$f(\rho) = H(1-\rho^2/L^2)$$

Uniform load … same solution as before.



Parabolic dome.

Non uniform load: load concentration in center part (lantern)

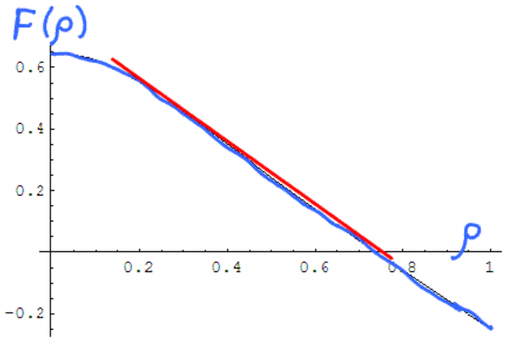


Transverse equilibrium

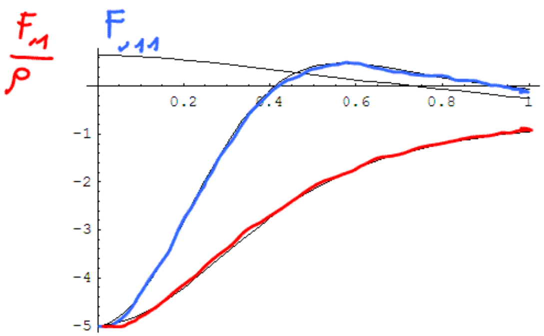
$$\frac{1}{\rho} F_{,1} f_{,11} + F_{,11} \frac{1}{\rho} f_{,1} - p = 0 \; .$$

Physical stress components:

$\sigma_{11} = F_{,1}/\rho$, $\sigma_{22} = F_{,11}$, $\sigma_{12} = 0$



Stress function

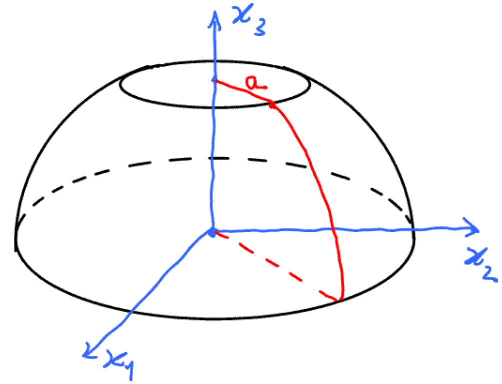


Stress

Central "eye"

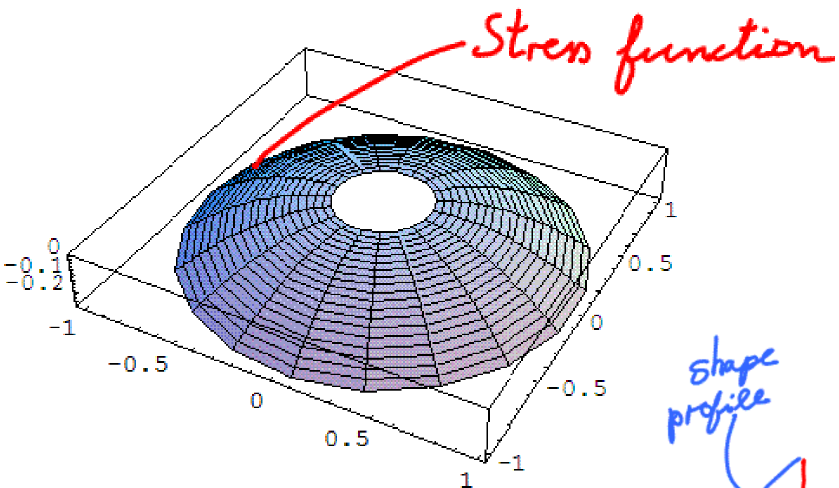
Uniform load.

Stress function: $F = -1/2\ \sigma\ (\rho - a)^2$



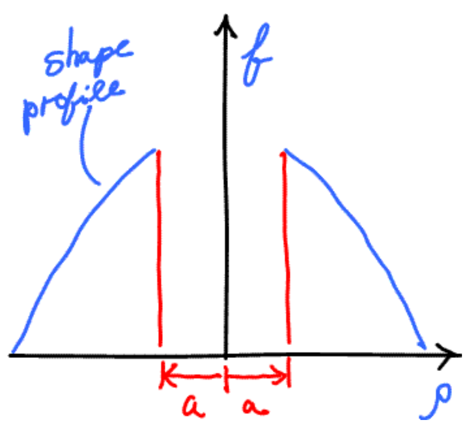
Stress:

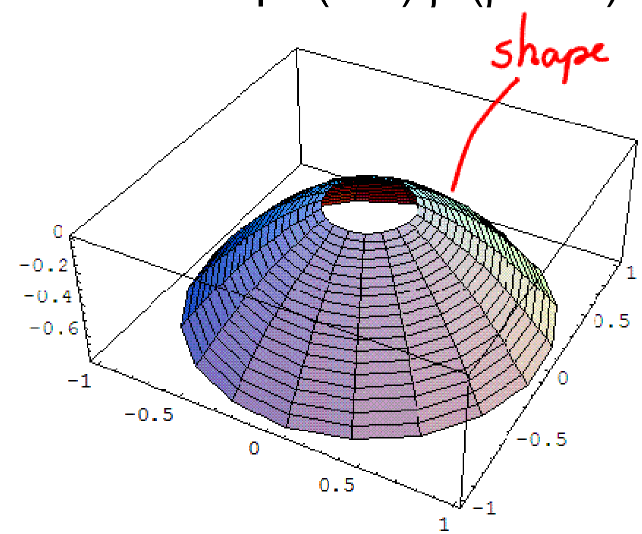
$\sigma_{11} = F_{,1}/\rho = -\ (\rho - a)\ \sigma/\rho\ ,\ \sigma_{22} = -\sigma\ ,\ \sigma_{12} = 0\ .$



Possible solution for the shape:

shape function: $f = -p°/(4\ \sigma)\ \rho\ (\rho + 2a)$





Helical stair

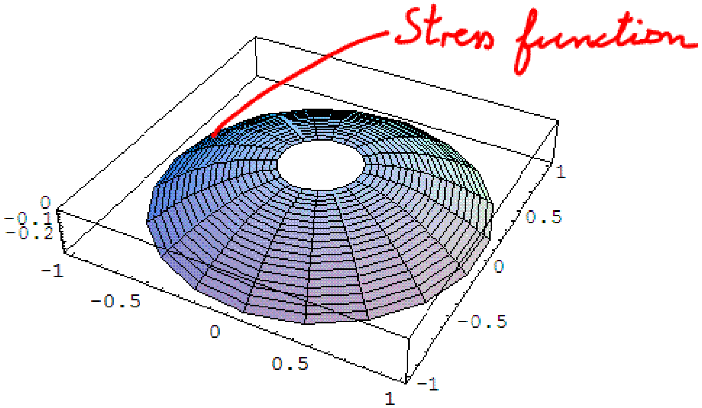
Uniform load: p°.   f depends on $\theta^2 = \theta$

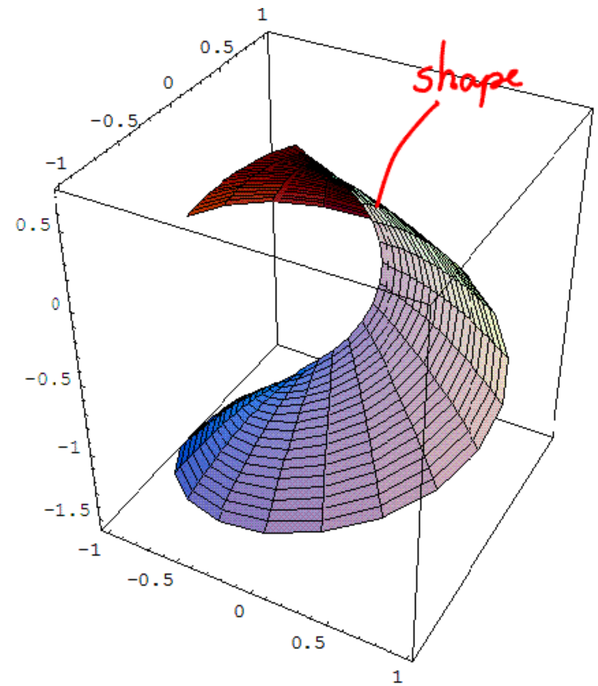
…For F use same stress function as before (independent of θ):

Stress function: $F = -1/2\ \sigma\ (\rho - a)^2$

Stress: $\sigma_{11} = F_{,1}/\rho = -(\rho - a)\ \sigma/\rho$, $\sigma_{22} = -\sigma$, $\sigma_{12} = 0$.

Transverse equilibrium: $\frac{1}{\rho} F_{,1} f_{,11} + F_{,11}\left(\frac{1}{\rho} f_{,1} + \frac{1}{\rho^2} f_{,22}\right) = p^\circ$.





Possible solution for the shape:

shape function: $f = -p^\circ/(4\ \sigma)\ \rho\ (\rho + 2a) + k\ \theta$